A Level

Article: 273

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A Level

Article No. 273

Mathematics 9709

Paper-3

(Topical Past Paper with Mark Scheme)

(2002-2019)

Features:

- ✓ All Variants
- ✓ Mark schemes included
- ✓ Questions order new to old

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Algebra

A-Level
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Topical Workbook

Topics

- **1.1 Equation & inequalities**
- **1.2 Remainder & factor theorem**
- **1.3 Partial fractions & binomial expansions**

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Unit-1: Algebra

1.1: Equation and Inequalities

1. **M/J 17/P32/Q2**
   Solve the inequality $|x-3| < 3x-4$. [4]

2. **M/J 17/P31/Q1**
   Solve the inequality $|2x+1| < 3|x-2|$. [4]

3. **M/J 16/P31/Q1**
   (i) Solve the equation $2|x-1| = 3|x|$. [3]
   (ii) Hence solve the equation $2|5^x - 1| = 3|5^x|$, giving your answer correct to 3 significant figures. [2]

4. **M/J 16/P33/Q1**
   Solve the inequality $2|x-2| > |3x+1|$. [4]

5. **O/N 15/P32/Q1, O/N 15/P31/Q1**
   Solve the inequality $|2x-5| > 3|2x+1|$. [4]

6. **M/J 15/P33/Q2**
   Solve the inequality $|x-2| > 2x-3$. [4]

7. **O/N 14/P33/Q1**
   Solve the inequality $|3x-1| < |2x+5|$. [4]

8. **M/J 14/P32/Q1**
   Find the set of values of x satisfying the inequality
   $$|x+2a| > 3|x-a|,$$
   where a is a positive constant. [4]

9. **M/J 13/P32/Q1**
   Solve the equation $|x-2| = |\frac{1}{3}x|$. [3]

10. **M/J 13/P31/Q3**
    Express $\dfrac{7x^2-3x+2}{x(x^2+1)}$ in partial fractions. [5]

11. **M/J 13/P33/Q1**
    Solve the inequality $|4x+3| > |x|$. [4]

12. **O/N 12/P32/Q1, O/N 12/P31/Q1**
    Find the set of values of x satisfying the inequality $3|x-1| < |2x+1|$. [4]

13. **M/J 12/P31/Q1**
    Solve the equation $|4-2^x| = 10$, giving your answer correct to 3 significant figures. [3]

14. **O/N 11/P31/Q3**
    The polynomial x^4+3x^3+ax+3 is denoted by $p(x)$. It is given that $p(x)$ is divisible by x^2-x+1.
    (i) Find the value of a. [4]
    (ii) When a has this value, find the real roots of the equation $p(x) = 0$. [2]

15. **M/J 11/P32/Q1**
    Solve the inequality $|x| < |5+2x|$. [3]

16. M/J 11/P31/Q4

The polynomial $f(x)$ is defined by

$$f(x) = 12x^3 + 25x^2 - 4x - 12.$$

(i) Show that $f(-2) = 0$ and factorise $f(x)$ completely. [4]

(ii) Given that

$$12 \times 27^y + 25 \times 9^y - 4 \times 3^y - 12 = 0,$$

state the value of 3^y and hence find y correct to 3 significant figures. [3]

17. M/J 11/P33/Q5

The polynomial $ax^3 + bx^2 + 5x - 2$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x - 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x - 2)$ the remainder is 12.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, find the quadratic factor of $p(x)$. [2]

18. O/N 10/P32/Q1, O/N 10/P31/Q1

Solve the inequality $2|x - 3| > |3x + 1|$. [4]

19. O/N 10/P33/Q10

The polynomial $p(z)$ is defined by

$$p(z) = z^3 + mz^2 + 24z + 32,$$

where m is a constant. It is given that $(z + 2)$ is a factor of $p(z)$.

(i) Find the value of m. [2]

(ii) Hence, showing all your working, find

(a) the three roots of the equation $p(z) = 0$, [5]

(b) the six roots of the equation $p(z^2) = 0$. [6]

20. M/J 10/P32/Q5

The polynomial $2x^3 + 5x^2 + ax + b$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x + 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x + 2)$ the remainder is 9.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, factorise $p(x)$ completely. [3]

21. M/J 10/P31/Q1

Solve the inequality $|x + 3a| > 2|x - 2a|$, where a is a positive constant. [4]

22. M/J 10/P33/Q1

Solve the inequality $|x - 3| > 2|x + 1|$. [4]

23. O/N 09/P31/Q1

Solve the inequality $2 - 3x < |x - 3|$. [4]

24. M/J 08/P03/Q1

Solve the inequality $|x - 2| > 3|2x + 1|$. [4]

25. O/N 06/P03/Q1

Find the set of values of x satisfying the inequality $|3^x - 8| < 0.5$, giving 3 significant figures in your answer. [4]

26. M/J 06/P03/Q2

Solve the inequality $2x > |x - 1|$. [4]

27. O/N 05/P03/Q1

Given that a is a positive constant, solve the inequality

$$|x - 3a| > |x - a|.$$ [4]

28. M/J 04/P03/Q2

Solve the inequality $|2x+1| < |x|$. [4]

29. O/N 03/P03/Q1

Solve the inequality $|2^x - 8| < 5$. [4]

30. M/J 03/P03/Q3

Solve the inequality $|x-2| < 3-2x$. [4]

31. O/N 02/P03/Q1

Solve the inequality $|9-2x| < 1$. [3]

Answers Section

1. M/J 17/P32/Q2

EITHER:

State or imply non-modular inequality $(x-3)^2 < (3x-4)^2$, or corresponding equation

Make reasonable attempt at solving a three term quadratic

Obtain critical value $x = \frac{7}{4}$

State final answer $x > \frac{7}{4}$ only

OR1:

State the relevant critical inequality $3-x < 3x-4$, or corresponding equation

Solve for x

Obtain critical value $x = \frac{7}{4}$

State final answer $x > \frac{7}{4}$ only

OR2:

Make recognizable sketches of $y = |x-3|$ and $y = 3x-4$ on a single diagram

Find x-coordinate of the intersection

Obtain $x = \frac{7}{4}$

State final answer $x > \frac{7}{4}$ only

4

2. M/J 17/P31/Q1

EITHER:

State or imply non-modular inequality $(2x+1)^2 < (3(x-2))^2$, or corresponding quadratic equation, or pair of linear equations $(2x+1) = \pm 3(x-2)$

Make reasonable solution attempt at a 3-term quadratic e.g. $5x^2 - 40x + 35 = 0$ or solve two linear equations for x

Obtain critical values $x = 1$ and $x = 7$

State final answer $x < 1$ and $x > 7$

OR:

Obtain critical value $x = 7$ from a graphical method, or by inspection, or by solving a linear equation or inequality

Obtain critical value $x = 1$ similarly

State final answer $x < 1$ and $x > 7$

3. M/J 16/P31/Q1

(i) *EITHER*: State or imply non-modular equation $(2(x-1))^2 = (3x)^2$, or pair of linear equations $2(x-1) = \pm 3x$

Make reasonable solution attempt at a 3-term quadratic, or solve two linear equations

Obtain answers $x = -2$ and $x = \frac{2}{5}$

OR: Obtain answer $x = -2$ by inspection or by solving a linear equation

Obtain answer $x = \frac{2}{5}$ similarly

(ii) Use correct method for solving an equation of the form $5^x = a$ or $5^{x+1} = a$, where $a > 0$

Obtain answer $x = -0.569$ only

[2]

4. **M/J 16/P33/Q1**

EITHER: State or imply non-modular inequality $(2(x-2))^2 > (3x+1)^2$, or corresponding quadratic equation, or pair of linear equations $2(x-2) = \pm(3x+1)$

Make reasonable solution attempt at a 3-term quadratic, or solve two linear equations for x

Obtain critical values $x=-5$ and $x=\frac{3}{5}$

State final answer $-5 < x < \frac{3}{5}$

OR: Obtain critical value $x=-5$ from a graphical method, or by inspection, or by solving a linear equation or inequality

Obtain critical value $x=\frac{3}{5}$ similarly

State final answer $-5 < x < \frac{3}{5}$ [4]

[Do not condone $\le$ for $<$.]

5. **O/N 15/P32/Q1, O/N 15/P31/Q1**

EITHER: State or imply non-modular inequality $(2x-5)^2 > (3(2x+1))^2$, or corresponding quadratic equation, or pair of linear equations $(2x-5) = \pm 3(2x+1)$

Make reasonable solution attempt at a 3-term quadratic, or solve two linear equations for x

Obtain critical values -2 and $\frac{1}{4}$

State final answer $-2 < x < \frac{1}{4}$

OR: Obtain critical value $x=-2$ from a graphical method, or by inspection, or by solving a linear equation or inequality

Obtain critical value $x=\frac{1}{4}$ similarly [4]

State final answer $-2 < x < \frac{1}{4}$

[Do not condone $\leqslant$ for $<$]

6. **M/J 15/P33/Q2**

EITHER: State or imply non-modular inequality $(x-2)^2 > (2x-3)^2$, or corresponding equation

Solve a 3-term quadratic, as in Q1.

Obtain critical value $x = \frac{5}{3}$

State final answer $x < \frac{5}{3}$ only

OR1: State the relevant critical linear inequality $(2-x) > (2x-3)$, or corresponding equation

Solve inequality or equation for x

Obtain critical value $x = \frac{5}{3}$

State final answer $x < \frac{5}{3}$ only

OR2: Make recognisable sketches of $y = 2x-3$ and $y = |x-2|$ on a single diagram

Find x-coordinate of the intersection

Obtain $x = \frac{5}{3}$

State final answer $x < \frac{5}{3}$ only 4

7. O/N 14/P33/Q1

Either State or imply non-modular inequality $(3x-1)^2<(2x+5)^2$ or corresponding quadratic equation or pair of linear equations $3x-1=\pm(2x+5)$

Solve a three-term quadratic or two linear equations $5x^2-26x-24<0$

Obtain $-\dfrac{4}{5}$ and 6

State $-\dfrac{4}{5}<x<6$

Or Obtain value 6 from graph, inspection or solving linear equation

Obtain value $-\dfrac{4}{5}$ similarly

State $-\dfrac{4}{5}<x<6$ [4]

8. M/J 14/P32/Q1

EITHER: State or imply non-modular inequality $(x+2a)^2>(3(x-a))^2$, or corresponding quadratic equation, or pair of linear equations $(x+2a)=\pm3(x-a)$

Make reasonable solution attempt at a 3-term quadratic, or solve two linear equations for x

Obtain critical values $x=\frac{1}{4}a$ and $x=\frac{5}{2}a$

State answer $\frac{1}{4}a<x<\frac{5}{2}a$

OR: Obtain critical value $x=\frac{5}{2}a$ from a graphical method, or by inspection, or by solving a linear equation or inequality

Obtain critical value $x=\frac{1}{4}a$ similarly

State answer $\frac{1}{4}a<x<\frac{5}{2}a$ 4

[Do not condone $\leqslant$ for $<$.]

9. M/J 13/P32/Q1

EITHER: State or imply non-modular equation $(x-2)^2=\left(\dfrac{1}{3}x\right)^2$,

or pair of equations $x-2=\pm\dfrac{1}{3}x$

Obtain answer $x=3$

Obtain answer $x=\dfrac{3}{2}$, or equivalent

OR: Obtain answer $x=3$ by solving an equation or by inspection

State or imply the equation $x-2=-\dfrac{1}{3}$, or equivalent

Obtain answer $x=\dfrac{3}{2}$, or equivalent [3]

10. M/J 13/P31/Q3

State or imply correct form $\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}$

Use any relevant method to find at least one constant

Obtain $A=2$

Obtain $B=5$

Obtain $C=-3$ [5]

11. M/J 13/P33/Q1

EITHER: State or imply non-modular inequality $(4x+3)^2 > x^2$, or corresponding equation or pair of equations $4x+3 = \pm x$

Obtain a critical value, e.g. -1

Obtain a second critical value, e.g. $-\frac{3}{5}$

State final answer $x < -1, x > -\frac{3}{5}$

OR: Obtain critical value $x = -1$, by solving a linear equation or inequality, or from a graphical method or by inspection

Obtain the critical value $-\frac{3}{5}$ similarly

State final answer $x < -1, x > -\frac{3}{5}$ [4]

[Do not condone $\le$ or $\ge$.]

12. O/N 12/P32/Q1, O/N 12/P31/Q1

EITHER State or imply non-modular inequality $(3(x-1))^2 < (2x+1)^2$ or corresponding quadratic equation, or pair of linear equations $3(x-1) = \pm(2x+1)$

Make reasonable solution attempt at a 3-term quadratic, or solve two linear equations

Obtain critical values $x = \frac{2}{5}$ and $x = 4$

State answer $\frac{2}{5} < x < 4$

OR Obtain critical value $x = \frac{2}{5}$ or $x = 4$ from a graphical method, or by inspection, or by solving a linear equation or inequality

Obtain critical values $x = \frac{2}{5}$ and $x = 4$

State answer $\frac{2}{5} < x < 4$ [4]

[Do not condone $\le$ for $<$.]

13. M/J 12/P31/Q1

State or imply $4 - 2^x = -10$ and 10

Use correct method for solving equation of form $2^x = a$

Obtain 3.81 [3]

14. O/N 11/P31/Q3

(i) *EITHER*: Attempt division by $x^2 - x + 1$ reaching a partial quotient of $x^2 + kx$

Obtain quotient $x^2 + 4x + 3$

Equate remainder of form lx to zero and solve for a, or equivalent

Obtain answer $a = 1$

OR: Substitute a complex zero of $x^2 - x + 1$ in $p(x)$ and equate to zero

Obtain a correct equation in a in any unsimplified form

Expand terms, use $i^2 = -1$ and solve for a

Obtain answer $a = 1$ [4]

[SR: The first M1 is earned if inspection reaches an unknown factor $x^2 + Bx + C$ and an equation in B and/or C, or an unknown factor $Ax^2 + Bx + 3$ and an equation in A and/or B. The second M1 is only earned if use of the equation $a = B - C$ is seen or implied.]

(ii) State answer, e.g. $x = -3$

State answer, e.g. $x = -1$ and no others [2]

15. M/J 11/P32/Q1

EITHER: State or imply non-modular inequality $x^2 < (5+2x)^2$, or corresponding equation, or pair of linear equations $x = \pm(5+2x)$

Obtain critical values -5 and $-\frac{5}{3}$ only

Obtain final answer $x < -5, x > -\frac{5}{3}$

OR: State one critical value e.g. -5, by solving a linear equation or inequality, or from a graphical method, or by inspection

State the other critical value, e.g. $-\frac{5}{3}$, and no other

Obtain final answer $x < -5, x > -\frac{5}{3}$ [3]

[Do not condone $\leq$ or $\geq$.]

16. M/J 11/P31/Q4

(i) Verify that $-96 + 100 + 8 - 12 = 0$

Attempt to find quadratic factor by division by $(x + 2)$, reaching a partial quotient $12x^2 + kx$, inspection or use of an identity

Obtain $12x^2 + x - 6$

State $(x + 2)(4x + 3)(3x - 2)$ [4]

[The M1 can be earned if inspection has unknown factor $Ax^2 + Bx - 6$ and an equation in A and/or B or equation $12x^2 + Bx + C$ and an equation in B and/or C.]

(ii) State $3^y = \frac{2}{3}$ and no other value

Use correct method for finding y from equation of form $3^y = k$, where $k > 0$

Obtain -0.369 and no other value [3]

17. M/J 11/P33/Q5

(i) Substitute $x = \frac{1}{2}$ and equate to zero, or divide, and obtain a correct equation, e.g. $\frac{1}{8}a + \frac{1}{4}b + \frac{5}{2} - 2 = 0$

Substitute $x = 2$ and equate result to 12, or divide and equate constant remainder to 12

Obtain a correct equation, e.g. $8a + 4b + 10 - 2 = 12$

Solve for a or for b

Obtain $a = 2$ and $b = -3$ [5]

(ii) Attempt division by $2x - 1$ reaching a partial quotient $\frac{1}{2}ax^2 + kx$

Obtain quadratic factor $x^2 - x + 2$ [2]

[The M1 is earned if inspection has an unknown factor $Ax^2 + Bx + 2$ and an equation in A and/or B, or an unknown factor of $\frac{1}{2}ax^2 + Bx + C$ and an equation in B and/or C.]

18. O/N 10/P32/Q1, O/N 10/P31/Q1

EITHER: State or imply non-modular inequality $(2(x-3))^2 > (3x+1)^2$, or corresponding quadratic equation, or pair of linear equations $2(x-3) = \pm(3x+1)$

Make reasonable solution attempt at a 3-term quadratic, or solve two linear equations

Obtain critical values $x = -7$ and $x = 1$

State answer $-7 < x < 1$

OR: Obtain critical value $x = -7$ or $x = 1$ from a graphical method, or by inspection, or by solving a linear equation or inequality

Obtain critical values $x = -7$ and $x = 1$

State answer $-7 < x < 1$ [4]

[Do not condone: ‹ for <.]

19. O/N 10/P33/Q10

(i) Attempt to solve for m the equation $p(-2) = 0$ or equivalent
Obtain $m = 6$ [2]

Alternative:
Attempt $p(z) \div (z + 2)$, equate a constant remainder to zero and solve for m.
Obtain $m = 6$

(ii) (a) State $z = -2$
Attempt to find quadratic factor by inspection, division, identity, …
Obtain $z^2 + 4z + 16$
Use correct method to solve a 3-term quadratic equation
Obtain $-2 \pm 2\sqrt{3}\mathrm{i}$ or equivalent [5]

(b) State or imply that square roots of answers from part (ii)(a) needed
Obtain $\pm \mathrm{i}\sqrt{2}$
Attempt to find square root of a further root in the form $x + \mathrm{i}y$ or in polar form
Obtain $a^2 - b^2 = -2$ and $ab = (\pm)\sqrt{3}$ following their answer to part (ii)(a)
Solve for a and b
Obtain $\pm\left(1 + \mathrm{i}\sqrt{3}\right)$ and $\pm\left(1 - \mathrm{i}\sqrt{3}\right)$ [6]

20. M/J 10/P32/Q5

(i) Substitute $x = -\frac{1}{2}$, equate to zero and obtain a correct equation, e.g.
$-\frac{1}{4} + \frac{5}{4} - \frac{1}{2}a + b = 0$
Substitute $x = -2$ and equate to 9
Obtain a correct equation, e.g. $-16 + 20 - 2a + b = 9$
Solve for a or for b
Obtain $a = -4$ and $b = -3$ [5]

(ii) Attempt division by $2x + 1$ reaching a partial quotient of $x^2 + kx$
Obtain quadratic factor $x^2 + 2x - 3$
Obtain factorisation $(2x + 1)(x + 3)(x - 1)$ [3]

[The M1 is earned if inspection has an unknown factor of $x^2 + ex + f$ and an equation in e and/or f, or if two coefficients with the correct moduli are stated without working.]
[If linear factors are found by the factor theorem, give B1 + B1 for $(x - 1)$ and $(x + 3)$, and then B1 for the complete factorisation.]

21. M/J 10/P31/Q1

EITHER: State or imply non-modular inequality $(x + 3a)^2 > (2(x - 2a))^2$, or corresponding quadratic equation, or pair of linear equations $(x + 3a) = \pm 2(x - 2a)$
Make reasonable solution attempt at a 3-term quadratic, or solve two linear equations
Obtain critical values $x = \frac{1}{3}a$ and $x = 7a$
State answer $\frac{1}{3}a < x < 7a$

OR: Obtain the critical value $x = 7a$ from a graphical method, or by inspection, or by solving a linear equation or inequality
Obtain the critical value $x = \frac{1}{3}a$ similarly
State answer $\frac{1}{3}a < x < 7a$ [4]
[Do not condone $\leqslant$ for $<$; accept 0.33 for $\frac{1}{3}$.]

22. M/J 10/P33/Q1

EITHER: State or imply non-modular inequality $(x-3)^2 > (2(x+1))^2$, or corresponding quadratic equation, or pair of linear equations $(x-3) = \pm 2(x+1)$
Make reasonable solution attempt at a 3-term quadratic, or solve two linear equations
Obtain critical values -5 and $\frac{1}{3}$
State answer $-5 < x < \frac{1}{3}$

OR: Obtain the critical value $x = -5$ from a graphical method, or by inspection, or by solving a linear equation or inequality
Obtain the critical value $x = \frac{1}{3}$ similarly
State answer $-5 < x < \frac{1}{3}$ [4]
[Do not condone ≤ for <; accept 0.33 for $\frac{1}{3}$.]

23. O/N 09/P31/Q1

EITHER: State or imply non-modular inequality $(2-3x)^2 < (x-3)^2$, or corresponding equation, and make a reasonable solution attempt at a 3-term quadratic
Obtain critical value $x = -\frac{1}{2}$
Obtain $x > -\frac{1}{2}$
Fully justify $x > -\frac{1}{2}$ as only answer

*OR*1: State the relevant critical linear equation, i.e. $2 - 3x = 3 - x$
Obtain critical value $x = -\frac{1}{2}$
Obtain $x > -\frac{1}{2}$
Fully justify $x > -\frac{1}{2}$ as only answer

*OR*2: Obtain the critical value $x = -\frac{1}{2}$ by inspection, or by solving a linear inequality
Obtain $x > -\frac{1}{2}$
Fully justify $x > -\frac{1}{2}$ as only answer

*OR*3: Make recognisable sketches of $y = 2 - 3x$ and $y = |x-3|$ on a single diagram
Obtain critical value $x = -\frac{1}{2}$
Obtain $x > -\frac{1}{2}$
Fully justify $x > -\frac{1}{2}$ as only answer [4]
[Condone ⩾ for > in the third mark but not the fourth.]

24. M/J 08/P03/Q1

EITHER State or imply non-modular inequality $(x-2)^2 > (3(2x+1))^2$, or corresponding quadratic equation, or pair of linear equations $(x-2) = \pm 3(2x+1)$
Make reasonable solution attempt at a 3-term quadratic, or solve two linear equations
Obtain critical values $x = -1$ and $x = -\frac{1}{7}$
State answer $-1 < x < -\frac{1}{7}$

OR Obtain the critical value $x = -1$ from a graphical method, or by inspection, or by solving a linear equation or inequality
Obtain the critical value $x = -\frac{1}{7}$ similarly
State answer $-1 < x < -\frac{1}{7}$ [4]
[Do not condone ≤ for <; accept $-\frac{5}{35}$ and -0.14 for $-\frac{1}{7}$.]

25. O/N 06/P03/Q1

EITHER: State or imply non-modular inequality $-0.5 < 3^x - 8 < 0.5$, or $(3^x - 8)^2 < (0.5)^2$, or corresponding pair of linear equations or quadratic equation
Use correct method for solving an equation of the form $3^x = a$, where $a > 0$
Obtain critical values 1.83 and 1.95, or exact equivalents
State correct answer $1.83 < x < 1.95$

OR: Use correct method for solving an equation of the form $3^x = a$, where $a > 0$
Obtain one critical value, e.g. 1.95, or exact equivalent
Obtain the other critical value 1.83, or exact equivalent
State correct answer $1.83 < x < 1.95$ [4]
[Do not condone $\leq$ of $<$. Allow final answer given in the form $1.83 < x$, (and) $x < 1.95$.]
[Exact equivalents must be in terms of In or logarithms to base 10.]
[SR: Solutions given as logarithms to base 3 can only earn M1 and B1 of the first scheme.]

26. M/J 06/P03/Q2

EITHER: State or imply non-modular inequality $(2x)^2 > (x-1)^2$, or corresponding equation
Expand and make a reasonable solution attempt at a 2- or 3-term quadratic
Obtain critical value $x = \frac{1}{3}$
Status answer $x > \frac{1}{3}$ only

OR: State the relevant critical linear equation, i.e. $2x = 1 - x$
Obtain critical value $x = \frac{1}{3}$
Status answer $x > \frac{1}{3}$ only
State or imply by omission that no other answer exists

OR: Obtain the critical value $x = \frac{1}{3}$ from a graphical method, or by inspection, or by solving a linear inequality
Status answer $x > \frac{1}{3}$
State or imply by omission that no other answer exists [4]

27. O/N 05/P03/Q1

EITHER : State or imply non-modular inequality $(x-3a)^2 > (x-a)^2$, or corresponding equation
Expand and solve the inequality, or equivalent
Obtain critical value $2a$
State correct answer $x < 2a$ only

OR : State a correct linear equation for the critical value, e.g. $x - 3a = -(x - a)$, or corresponding inequality
Solve the linear equation for x, or equivalent
Obtain critical value $2a$
State correct answer $x < 2a$ only

OR : Make recognizable sketches of both $y = |x - 3a|$ and $y = |x - a|$ on a single diagram
Obtain a critical value from the intersection of the graphs
Obtain critical value $2a$
Obtain correct answer $x < 2a$ only [4]

28. M/J 04/P03/Q2

EITHER: State or imply non-modular inequality $(2x+1)^2 < x^2$ or corresponding quadratic equation or pair of linear equations $(2x + 1) = \pm x$
Expand and make a reasonable solution attempt at a 3-term quadratic, or solve two linear equations
Obtain critical values $x = -1$ and $x = -\frac{1}{3}$
State answer $-1 < x < -\frac{1}{3}$

OR: Obtain the critical value $x = -1$ from a graphical method , or by inspection, or by solving a linear inequality or equation
Obtain the critical value $x = -\frac{1}{3}$ (deduct B1 from B3 if extra values are obtained)
State answer $-1 < x < -\frac{1}{3}$ **4**
[Condone ≤ for <; accept −0.33 for $-\frac{1}{3}$.]

29. O/N 03/P03/Q1

EITHER: State or imply non-modular inequality $-5 < 2^x - 8 < 5$, or $(2^x - 8)^2 < 5^2$ or corresponding pair of linear equations or quadratic equation
Use correct method for solving an equation of the form $2^x = a$
Obtain critical values 1.58 and 3.70, or exact equivalents
State correct answer $1.58 < x < 3.70$

OR: Use correct method for solving an equation of the form $2^x = a$
Obtain one critical value (probably 3.70), or exact equivalent
Obtain the other critical value, or exact equivalent
State correct answer $1.58 < x < 3.70$ **[4]**

[Allow 1.59 and 3.7. Condone ≤ for <. Allow final answers given separately. Exact equivalents must be in terms of ln or logarithms to base 10.]
[SR: Solutions given as logarithms to base 2 can only earn M1 and B1 of the first scheme.]

30. M/J 03/P03/Q3

EITHER State or imply non-modular inequality $(x - 2)^2 < (3 - 2x)^2$, or corresponding equation
Expand and make a reasonable solution attempt at a 2- or 3-term quadratic, or equivalent
Obtain critical value $x = 1$
State answer $x < 1$ only

OR State the relevant linear equation for a critical value, i.e. $2 - x = 3 - 2x$, or equivalent
Obtain critical value $x = 1$
State answer $x < 1$
State or imply by omission that no other answer exists

OR Obtain the critical value $x = 1$ from a graphical method, or by inspection, or by solving a linear inequality
State answer $x < 1$
State or imply by omission that no other answer exists **[4]**

31. O/N 02/P03/Q1

EITHER: State or imply non-modular inequality $(9 - 2x)^2 < 1$, or a correct pair of linear inequalities, combined or separate, e.g. $-1 < 9 - 2x < 1$
Obtain both critical values 4 and 5
State correct answer $4 < x < 5$; accept $x > 4$, $x < 5$

OR: State a correct equation or pair of equations for both criticle values, e.g. $9 - 2x = 1$ and $9 - 2x = -1$ or $9 - 2x = \pm 1$
Obtain critical values 4 and 5

OR: State one critical value (probably $= 4$) from a graphical method or by inspection or by solving a linear inequality or equation
State the other critical value correctly
State correct answer $4 < x < 5$; accept $x > 4$, $x < 5$ [3]
[use of ≤, throughout, or at the end, scores a maximum of B2.]

1.2: Remainder and Factor Theorem

1. **M/J 18/P31/Q4**
   The polynomial $x^4 + 2x^3 + ax + b$, where a and b are constants, is divisible by $x^2 - x + 1$. Find the values of a and b. [5]

2. **O/N 17/P31/Q1, O/N 17/P33/Q1**
   Find the quotient and remainder when x^4 is divided by $x^2 + 2x - 1$. [3]

3. **O/N 16/P33/Q4**
   The polynomial $4x^4 + ax^2 + 11x + b$, where a and b are constants, is denoted by $\mathrm{p}(x)$. It is given that $\mathrm{p}(x)$ is divisible by $x^2 - x + 2$.
   (i) Find the values of a and b. [5]
   (ii) When a and b have these values, find the real roots of the equation $\mathrm{p}(x) = 0$. [2]

4. **O/N 15/P32/Q6, O/N 15/P31/Q6**
   The polynomial $8x^3 + ax^2 + bx - 1$, where a and b are constants, is denoted by $\mathrm{p}(x)$. It is given that $(x + 1)$ is a factor of $\mathrm{p}(x)$ and that when $\mathrm{p}(x)$ is divided by $(2x + 1)$ the remainder is 1.
   (i) Find the values of a and b. [5]
   (ii) When a and b have these values, factorise $\mathrm{p}(x)$ completely. [3]

5. **O/N 14/P31/Q3**
   The polynomial $ax^3 + bx^2 + x + 3$, where a and b are constants, is denoted by $\mathrm{p}(x)$. It is given that $(3x + 1)$ is a factor of $\mathrm{p}(x)$, and that when $\mathrm{p}(x)$ is divided by $(x - 2)$ the remainder is 21. Find the values of a and b. [5]

6. **O/N 14/P33/Q3**
   The polynomial $4x^3 + ax^2 + bx - 2$, where a and b are constants, is denoted by $\mathrm{p}(x)$. It is given that $(x + 1)$ and $(x + 2)$ are factors of $\mathrm{p}(x)$.
   (i) Find the values of a and b. [4]
   (ii) When a and b have these values, find the remainder when $\mathrm{p}(x)$ is divided by $(x^2 + 1)$. [3]

7. **M/J 14/P32/Q5**
   (i) The polynomial $\mathrm{f}(x)$ is of the form $(x - 2)^2\mathrm{g}(x)$, where $\mathrm{g}(x)$ is another polynomial. Show that $(x - 2)$ is a factor of $\mathrm{f}'(x)$. [2]
   (ii) The polynomial $x^5 + ax^4 + 3x^3 + bx^2 + a$, where a and b are constants, has a factor $(x - 2)^2$. Using the factor theorem and the result of part **(i)**, or otherwise, find the values of a and b. [5]

8. **M/J 14/P31/Q6**
   It is given that $2\ln(4x - 5) + \ln(x + 1) = 3\ln 3$.
   (i) Show that $16x^3 - 24x^2 - 15x - 2 = 0$. [3]
   (ii) By first using the factor theorem, factorise $16x^3 - 24x^2 - 15x - 2$ completely. [4]
   (iii) Hence solve the equation $2\ln(4x - 5) + \ln(x + 1) = 3\ln 3$. [1]

9. **O/N 13/P33/Q3**
   The polynomial $\mathrm{f}(x)$ is defined by
   $$\mathrm{f}(x) = x^3 + ax^2 - ax + 14,$$
   where a is a constant. It is given that $(x + 2)$ is a factor of $\mathrm{f}(x)$.
   (i) Find the value of a. [2]
   (ii) Show that, when a has this value, the equation $\mathrm{f}(x) = 0$ has only one real root. [3]

10. M/J 13/P32/Q4

The polynomial $ax^3 - 20x^2 + x + 3$, where a is a constant, is denoted by $p(x)$. It is given that $(3x + 1)$ is a factor of $p(x)$.

(i) Find the value of a. [3]

(ii) When a has this value, factorise $p(x)$ completely. [3]

11. M/J 13/P33/Q5

The polynomial $8x^3 + ax^2 + bx + 3$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x + 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(2x - 1)$ the remainder is 1.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, find the remainder when $p(x)$ is divided by $2x^2 - 1$. [3]

12. M/J 12/P31/Q3

The polynomial $p(x)$ is defined by

$$p(x) = x^3 - 3ax + 4a,$$

where a is a constant.

(i) Given that $(x - 2)$ is a factor of $p(x)$, find the value of a. [2]

(ii) When a has this value,

(a) factorise $p(x)$ completely, [3]

(b) find all the roots of the equation $p(x^2) = 0$. [2]

13. O/N 11/P33/Q7

The polynomial $p(x)$ is defined by

$$p(x) = ax^3 - x^2 + 4x - a,$$

where a is a constant. It is given that $(2x - 1)$ is a factor of $p(x)$.

(i) Find the value of a and hence factorise $p(x)$. [4]

(ii) When a has the value found in part (i), express $\frac{8x - 13}{p(x)}$ in partial fractions. [5]

14. O/N 09/P32/Q5

The polynomial $2x^3 + ax^2 + bx - 4$, where a and b are constants, is denoted by $p(x)$. The result of differentiating $p(x)$ with respect to x is denoted by $p'(x)$. It is given that $(x + 2)$ is a factor of $p(x)$ and of $p'(x)$.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, factorise $p(x)$ completely. [3]

15. O/N 08/P03/Q5

The polynomial $4x^3 - 4x^2 + 3x + a$, where a is a constant, is denoted by $p(x)$. It is given that $p(x)$ is divisible by $2x^2 - 3x + 3$.

(i) Find the value of a. [3]

(ii) When a has this value, solve the inequality $p(x) < 0$, justifying your answer. [3]

16. O/N 07/P03/Q2

The polynomial $x^4 + 3x^2 + a$, where a is a constant, is denoted by $p(x)$. It is given that $x^2 + x + 2$ is a factor of $p(x)$. Find the value of a and the other quadratic factor of $p(x)$. [4]

17. M/J 07/P03/Q2

The polynomial $x^3 - 2x + a$, where a is a constant, is denoted by $p(x)$. It is given that $(x + 2)$ is a factor of $p(x)$.

(i) Find the value of a. [2]

(ii) When a has this value, find the quadratic factor of $p(x)$. [2]

18. M/J 05/P03/Q5

The polynomial $x^4 + 5x + a$ is denoted by $\text{p}(x)$. It is given that $x^2 - x + 3$ is a factor of $\text{p}(x)$.

(i) Find the value of a and factorise $\text{p}(x)$ completely. [6]

(ii) Hence state the number of real roots of the equation $\text{p}(x) = 0$, justifying your answer. [2]

19. M/J 03/P03/Q4

The polynomial $x^4 - 2x^3 - 2x^2 + a$ is denoted by $\text{f}(x)$. It is given that $\text{f}(x)$ is divisible by $x^2 - 4x + 4$.

(i) Find the value of a. [3]

(ii) When a has this value, show that $\text{f}(x)$ is never negative. [4]

20. M/J 02/P03/Q3

The polynomial $x^4 + 4x^2 + x + a$ is denoted by $\text{p}(x)$. It is given that $(x^2 + x + 2)$ is a factor of $\text{p}(x)$.

Find the value of a and the other quadratic factor of $\text{p}(x)$. [4]

Answers Section

1. M/J 18/P31/Q4

EITHER: Commence division by $x^2 - x + 1$ and reach a partial quotient of the form $x^2 + kx$

Obtain quotient $x^2 + 3x + 2$

Either Set remainder identically equal to zero and solve for a or for b, *or* multiply given divisor and found quotient and obtain a or b

Obtain $a = 1$

Obtain $b = 2$

OR: Assume an unknown factor $x^2 + Bx + C$ and obtain an equation in B and/or C

Obtain $B = 3$ and $A = 2$

Either Use equations to obtain a or b *or* multiply given divisor and found factor to obtain a or b

Obtain $a = 1$

Obtain $b = 2$ [5]

2. O/N 17/P31/Q1, O/N 17/P33/Q1

Commence division and reach a partial quotient $x^2 + kx$

Obtain quotient $x^2 - 2x + 5$

Obtain remainder $-12x + 5$ [3]

3. O/N 16/P33/Q4

(i) Commence division by $x^2 - x + 2$ and reach a partial quotient $4x^2 + kx$

Obtain quotient $4x^2 + 4x + a - 4$ or $4x^2 + 4x + b/2$

Equate x or constant term to zero and solve for a or b

Obtain $a = 1$

Obtain $b = -6$ [5]

(ii) Show that $x^2 - x + 2 = 0$ has no real roots

Obtain roots $\frac{1}{2}$ and $-\frac{3}{2}$ from $4x^2 + 4x - 3 = 0$ [2]

4. O/N 15/P32/Q6, O/N 15/P31/Q6

(i) Substitute $x = -1$, equate to zero and simplify at least as far as $-8 + a - b - 1 = 0$

Substitute $x = -\frac{1}{2}$ and equate the result to 1

Obtain a correct equation in any form, e.g. $-1 + \frac{1}{4}a - \frac{1}{2}b - 1 = 1$

Solve for a or for b

Obtain $a = 6$ and $b = -3$ [5]

(ii) Commence division by $(x + 1)$ reaching a partial quotient $8x^2 + kx$

Obtain quadratic factor $8x^2 - 2x - 1$

Obtain factorisation $(x + 1)(4x + 1)(2x - 1)$ [3]

[The M1 is earned if inspection reaches an unknown factor $8x^2 + Bx + C$ and an equation in B and/or C, or an unknown factor $Ax^2 + Bx - 1$ and an equation in A and/or B.]
[If linear factors are found by the factor theorem, give B1B1 for $(2x - 1)$ and $(4x + 1)$, and B1 for the complete factorisation.]

5. **O/N 14/P31/Q3**

Substitute $x=-\frac{1}{3}$, equate result to zero or divide by $3x+1$ and equate the remainder to zero and obtain a correct equation, e.g. $-\frac{1}{27}a+\frac{1}{9}b-\frac{1}{3}+3=0$

Substitute $x=2$ and equate result to 21 or divide by $x-2$ and equate constant remainder to 21

Obtain a correct equation, e.g. $8a+4b+5=21$

Solve for a or for b

Obtain $a=12$ and $b=-20$ [5]

6. **O/N 14/P33/Q3**

(i) Either Equate $p(-1)$ or $p(-2)$ to zero or divide by $(x+1)$ or $(x+2)$ and equate constant remainder to zero.

Obtain two equations $a-b=6$ and $4a-2b=34$ or equivalents

Solve pair of equations for a or b

Obtain $a=11$ and $b=5$

Or State or imply third factor is $4x-1$

Carry out complete expansion of $(x+1)(x+2)(4x-1)$ or $(x+1)(x+2)(Cx+D)$

Obtain $a=11$

Obtain $b=5$ [4]

(ii) Use division or equivalent and obtaining linear remainder

Obtain quotient $4x+a$, following their value of a

Indicate remainder $x-13$ [3]

7. **M/J 14/P32/Q5**

(i) Differentiate f(x) and obtain $f'(x)=(x-2)^2g'(x)+2(x-2)g(x)$

Conclude that $(x-2)$ is a factor of $f'(x)$ [2]

(ii) *EITHER:* Substitute $x=2$, equate to zero and state a correct equation, e.g. $32+16a+24+4b+a=0$

Differentiate polynomial, substitute $x = 2$ and equate to zero or divide by $(x-2)$ and equate constant remainder to zero

Obtain a correct equation, e.g. $80+32a+36+4b=0$

OR1: Identify given polynomial with $(x-2)^2(x^3+Ax^2+Bx+C)$ and obtain an equation in a and/or b

Obtain a correct equation, e.g. $\frac{1}{4}a-4(4+a)+4=3$

Obtain a second correct equation, e.g. $-\frac{3}{4}a+4(4+a)=b$

OR2: Divide given polynomial by $(x-2)^2$ and obtain an equation in a and b

Obtain a correct equation, e.g. $29+8a+b+0$

Obtain a second correct equation, e.g. $176+47a+4b=0$

Solve for a or for b

Obtain $a=-4$ and $b=3$ [5]

8. **M/J 14/P31/Q6**

(i) Use law for the logarithm for a product or quotient or exponentiation AND for a power

Obtain $(4x-5)^2(x+1)=27$

Obtain given equation correctly $16x^3-24x^2-15x-2=0$ [3]

(ii) Obtain $x = 2$ is root or $(x-2)$ is a factor, or likewise with $x = -\frac{1}{4}$
Divide by $(x-2)$ to reach a quotient of the form $16x^2 + kx$
Obtain quotient $16x^2 + 8x + 1$
Obtain $(x-2)(4x+1)^2$ or $(x-2)$, $(4x+1)$, $(4x+1)$ [4]

(iii) State $x = 2$ only [1]

9. O/N 13/P33/Q3

(i) Substitute –2 and equate to zero or divide by $x + 2$ and equate remainder to zero or use –2 in synthetic division
Obtain $a = -1$ [2]

(ii) Attempt to find quadratic factor by division reaching $x^2 + kx$, or inspection as far as $(x+2)(x^2+Bx+c)$ and equations for one or both of B and C, or $(x+2)(Ax^2+Bx+7)$ and equations for one or both of A and B.
Obtain $x^2 - 3x + 7$
Use discriminant to obtain –19, or equivalent, and **confirm one root** cwo [3]

10. M/J 13/P32/Q4

(i) Substitute $x = -\frac{1}{3}$, or divide by $3x+1$, and obtain a correct equation,
e.g. $-\frac{1}{27}a - \frac{20}{9} - \frac{1}{3} + 3 = 0$
Solve for a an equation obtained by a valid method
Obtain $a = 12$ [3]

(ii) Commence division by $3x+1$ reaching a partial quotient $\frac{1}{3}ax^2 + kx$
Obtain quadratic factor $4x^2 - 8x + 3$
Obtain factorisation $(3x+1)(2x-1)(2x-3)$ [3]
[The M1 is earned if inspection reaches an unknown factor $\frac{1}{3}ax^2 + Bx + C$ and an equation in B and/or C, or an unknown factor $Ax^2 + Bx + 3$ and an equation in A and/or B, or if two coefficients with the correct moduli are stated without working.]
[If linear factors are found by the factor theorem, give B1B1 for $(2x-1)$ and $(2x-3)$, and B1 for the complete factorisation.]
[Synthetic division giving $12x^2 - 24x + 9$ as quadratic factor earns M1A1, but the final factorisation needs $(x + \frac{1}{3})$, or equivalent, in order to earn the second A1.]
[SR: If $x = \frac{1}{3}$ is used in substitution or synthetic division, give the M1 in part (i) but give M0 in part **(ii)**.]

11. M/J 13/P33/Q5

(i) Substitute $x = -\frac{1}{2}$, or divide by $(2x+1)$, and obtain a correct equation, e.g. $a - 2b + 8 = 0$
Substitute $x = \frac{1}{2}$ and equate to 1, or divide by $(2x-1)$ and equate constant remainder to 1
Obtain a correct equation, e.g. $a + 2b + 12 = 0$
Solve for a or for b
Obtain $a = -10$ and $b = -1$ [5]

(ii) Divide by $2x^2 - 1$ and reach a quotient of the form $4x + k$
Obtain quotient $4x - 5$
Obtain remainder $3x - 2$ [3]

12. M/J 12/P31/Q3

(i) Substitute $x = 2$ and equate to zero, or divide by $x - 2$ and equate constant remainder to zero, or equivalent
Obtain $a = 4$ [2]

(ii) (a) Find further (quadratic or linear) factor by division, inspection or factor theorem or equivalent
Obtain $x^2 + 2x - 8$ or $x + 4$
State $(x-2)^2(x+4)$ or equivalent [3]

(b) State any two of the four (or six) roots
State all roots ($\pm\sqrt{2}$, $\pm 2i$), provided two are purely imaginary [2]

13. O/N 11/P33/Q7

(i) Substitute $x = \frac{1}{2}$ and equate to zero
or divide by $(2x-1)$, reach $\frac{a}{2}x^2 + kx + \ldots$and equate remainder to zero
or by inspection reach $\frac{a}{2}x^2 + bx + c$ and an equation in b/c
or by inspection reach $Ax^2 + Bx + a$ and an equation in A/B
Obtain $a = 2$
Attempt to find quadratic factor by division or inspection or equivalent
Obtain $(2x-1)(x^2+2)$

(ii) State or imply form $\frac{A}{2x-1} + \frac{Bx+C}{x^2+2}$, following factors from part (i)
Use relevant method to find a constant
Obtain $A = -4$, following factors from part (i)
Obtain $B = 2$
Obtain $C = 5$ [4]

14. O/N 09/P32/Q5

(i) Substitute $x = -2$, equate to zero and state a correct equation, e.g. $-16 + 4a - 2b - 4 = 0$
Differentiate $p(x)$, substitute $x = -2$ and equate to zero
Obtain a correct equation, e.g. $24 - 4a + b = 0$
Solve for a or for b
Obtian $a = 7$ and $b = 4$ [5]

(ii) *EITHER*: State or imply $(x+2)^2$ is a factor
Attempt division by $(x+2)^2$ reaching a quotient $2x + k$ or use inspection with unknown factor $cx + d$ reaching $c = 2$ or $d = -1$
Obtain factorisation $(x+2)^2(2x-1)$

OR: Attempt division by $(x+2)$
Obtain quadratic factor $2x^2 + 3x - 2$
Obtain factorisation $(x+2)(x+2)(2x-1)$ [3]

[The M1 is earned if division reaches a partial quotient of $2x^2 + kx$, or if inspection has an unknown factor of $2x^2 + ex + f$ and an equation in e and/or f, or if two coefficients with the correct moduli are stated without working.]

15. O/N 08/P03/Q5

(i) *EITHER*: Attempt division by $2x^2 - 3x + 3$ and state partial quotient $2x$
Complete division and form an equation for a
Obtain $a = 3$

*OR*1: By inspection or using an unknown factor $bx + c$, obtain $b = 2$
Complete the factorisation and obtain a
Obtain $a = 3$

OR2: Find a complex root of $2x^2-3x+3=0$ and substitute it in p(x)
Equate a correct expression to zero
Obtain $a=3$

OR3: Use $2x^2 \equiv 3x-3$ in p(x) at least once
Reduce the expression to the form $a+c=0$, or equivalent
Obtain $a=3$ [3]

(ii) State answer $x<-\frac{1}{2}$ only
Carry out a complete method for showing $2x^2-3x+3$ is never zero
Complete the justification of the answer by showing that $2x^2-3x+3>0$ for all x [3]
[These last two marks are independent of the B mark, so B0M1A1 is possible. Alternative methods include (a) Complete the square M1 and use a correct completion to justify the answer A1; (b) Draw a recognizable graph of $y=2x^2+3x-3$ or p(x) M1 and use a correct graph to justify the answer A1; (c) Find the x-coordinate of the stationary point of $y=2x^2+3x-3$ and either find its y-coordinate or determine its nature M1, then use minimum point with correct coordinates to justify the answer A1.]
[Do not accept ≤ for <]

16. O/N 07/P03/Q2

EITHER: Attempt division by x^2+x+2 reaching a partial quotient of x^2+kx
Complete the division and obtain quotient x^2-x+2
Equate constant remainder to zero and solve for a
Obtain answer $a=4$

OR: Calling the unknown factor x^2+bx+c, obtain an equation in b and/or c, or state without working two coefficients with the correct moduli
Obtain factor x^2-x+2
Use $a=2c$ to find a
Obtain answer $a=4$ [4]

17. M/J 07/P03/Q2

(i) Substitute $x=-2$ and equate to zero, or divide by $x+2$ and equate constant remainder to zero, or use a factor Ax^2+Bx+C and reach an equation in a
Obtain answer $a=4$ [2]

(ii) Attempt to find quadratic factor by division or inspection
State or exhibit quadratic factor x^2-2x+2 [2]

[The M1 is earned if division reaches a partial quotient x^2+kx, or if inspection has an unknown factor x^2+bx+c and an equation in b and/or c, or if inspection without working states two coefficients with the correct moduli.]

18. M/J 05/P03/Q5

(i) **EITHER:** Attempt division by x^2-x+3 reaching a partial quotient x^2+kx
Complete division and equate constant remainder to zero
Obtain answer $a=-6$

OR: Commence inspection and reach unknown factor x^2+bx+c
Obtain $3c=a$ and an equation in c
Obtain answer $a=-6$

State or obtain factor x^2+x-2
State or obtain factors $x+2$ and $x-1$ [6]

(ii) State that $x^2+x-2=0$, has two (real) roots
Show that $x^2-x+3=0$, has no (real) roots [2]

19. M/J 03/P03/Q4

(i) *EITHER* State or imply that $x - 2$ is a factor of $f(x)$
Substitute 2 for x and equate to zero
Obtain answer $a = 8$

[The statement $(x-2)^2 = x^2 - 4x + 4$ earns B1.]

OR Commence division by $x^2 - 4x + 4$ and obtain partial quotient $x^2 + 2x$
Complete the division and equate the remainder to zero
Obtain answer $a = 8$

OR Commence inspection and obtain unknown factor $x^2 + 2x + c$
Obtain $4c = a$ and an equation in c
Obtain answer $a = 8$ **[3]**

(ii) *EITHER* Substitute $a = 8$ and find other factor $x^2 + 2x + 2$ by inspection or division
State that $x^2 - 4x + 4 \geq 0$ for all x (condone > for ≥)
Attempt to establish sign of the other factor
Show that $x^2 + 2x + 2 > 0$ for all x and complete the proof
[An attempt to find the zeros of the other factor earns M1.]

OR Equate derivative to zero and attempt to solve for x
Obtain $x = -\frac{1}{2}$ and 2
Show correctly that $f(x)$ has a minimum at each of these values
Having also obtained and considered $x = 0$, complete the proof **[4]**

20. M/J 02/P03/Q3

Attempt to find a and/or quadratic factor by division or by inspection
Obtain partial quotient or factor $x^2 - x$
State answer $a = 6$
State or imply the other factor is $x^2 - x + 3$ [4]

[The M1 is earned if division has produced a partial quotient $x^2 + bx$, or if inspection has an unknown Factor $x^2 + bx + c$ and has reached an equation in b and/or c.]
[SR: a correct division with unresolved constant remainder can earn M1A1B0A1.]
[NB: successive division by a pair of incorrect linear factors, e.g. $x - 1$ and $x + 2$ or $x + 1$ and $x + 2$, can earn M1A0 or M1A1 (if their product is of the form $x^2 + x + k$).]

1.3: Partial Fractions and Binomial Expansions

1. M/J 18/P32/Q9

Let $f(x) = \dfrac{x - 4x^2}{(3-x)(2+x^2)}$.

(i) Express $f(x)$ in the form $\dfrac{A}{3-x} + \dfrac{Bx+C}{2+x^2}$. [4]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. [5]

2. M/J 18/P31/Q9

Let $f(x) = \dfrac{12x^2 + 4x - 1}{(x-1)(3x+2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

3. M/J 18/P33/Q1

Expand $\dfrac{4}{\sqrt{(4-3x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

4. O/N 17/P32/Q8

Let $f(x) = \dfrac{8x^2 + 9x + 8}{(1-x)(2x+3)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

5. M/J 17/P32/Q8

Let $f(x) = \dfrac{5x^2 - 7x + 4}{(3x+2)(x^2+5)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

6. M/J 17/P31/Q2

Expand $\dfrac{1}{\sqrt[3]{(1+6x)}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

7. M/J 17/P33/Q2

Expand $(3+2x)^{-3}$ in ascending powers of x up to and including the term in x^2, simplifying the coefficients. [4]

8. O/N 16/P32/Q2, O/N 16/P31/Q2

Expand $(2-x)(1+2x)^{-\frac{3}{2}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

9. O/N 16/P33/Q8

Let $f(x) = \dfrac{3x^2 + x + 6}{(x+2)(x^2+4)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

10. M/J 16/P32/Q2

Expand $\dfrac{1}{\sqrt{(1-2x)}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

11. M/J 16/P31/Q8

Let $f(x) = \dfrac{4x^2+12}{(x+1)(x-3)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

12. M/J 16/P33/Q10

Let $f(x) = \dfrac{10x-2x^2}{(x+3)(x-1)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

13. O/N 15/P33/Q2

Given that $\sqrt[3]{(1+9x)} \approx 1 + 3x + ax^2 + bx^3$ for small values of x, find the values of the coefficients a and b. [3]

14. M/J 15/P32/Q8

Let $f(x) = \dfrac{5x^2+x+6}{(3-2x)(x^2+4)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

15. M/J 15/P31/Q3

Show that, for small values of x^2,

$$(1-2x^2)^{-2} - (1+6x^2)^{\frac{2}{3}} \approx kx^4,$$

where the value of the constant k is to be determined. [6]

16. O/N 14/P32/Q3

The polynomial $ax^3 + bx^2 + x + 3$, where a and b are constants, is denoted by $p(x)$. It is given that $(3x+1)$ is a factor of $p(x)$, and that when $p(x)$ is divided by $(x-2)$ the remainder is 21. Find the values of a and b. [5]

17. O/N 14/P32/Q9, O/N 14/P31/Q9

Let $f(x) = \dfrac{x^2-8x+9}{(1-x)(2-x)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

18. M/J 14/P31/Q9

(i) Express $\dfrac{4+12x+x^2}{(3-x)(1+2x)^2}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{4+12x+x^2}{(3-x)(1+2x)^2}$ in ascending powers of x, up to and including the term in x^2. [5]

19. M/J 14/P33/Q2

Expand $(1+3x)^{-\frac{1}{3}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

20. O/N 13/P32/Q7

Let $f(x) = \dfrac{2x^2 - 7x - 1}{(x-2)(x^2+3)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

21. O/N 13/P33/Q8

(i) Express $\dfrac{7x^2+8}{(1+x)^2(2-3x)}$ in partial fractions. [5]

(ii) Hence expand $\dfrac{7x^2+8}{(1+x)^2(2-3x)}$ in ascending powers of x up to and including the term in x^2, simplifying the coefficients. [5]

22. M/J 13/P32/Q8(i)

(i) Express $\dfrac{1}{x^2(2x+1)}$ in the form $\dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{C}{2x+1}$. [4]

23. M/J 13/P31/Q1

Find the quotient and remainder when $2x^2$ is divided by $x+2$. [3]

24. M/J 13/P31/Q2

Expand $\dfrac{1+3x}{\sqrt{(1+2x)}}$ in ascending powers of x up to and including the term in x^2, simplifying the coefficients. [4]

25. O/N 12/P32/Q4, O/N 12/P31/Q4

When $(1+ax)^{-2}$, where a is a positive constant, is expanded in ascending powers of x, the coefficients of x and x^3 are equal.

(i) Find the exact value of a. [4]

(ii) When a has this value, obtain the expansion up to and including the term in x^2, simplifying the coefficients. [3]

26. O/N 12/P33/Q9

(i) Express $\dfrac{9-7x+8x^2}{(3-x)(1+x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{9-7x+8x^2}{(3-x)(1+x^2)}$ in ascending powers of x, up to and including the term in x^3. [5]

27. M/J 12/P32/Q3

Expand $\sqrt{\left(\dfrac{1-x}{1+x}\right)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [5]

28. M/J 12/P31/Q2

(i) Expand $\dfrac{1}{\sqrt{(1-4x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [3]

(ii) Hence find the coefficient of x^2 in the expansion of $\dfrac{1+2x}{\sqrt{(4-16x)}}$. [2]

29. M/J 12/P33/Q1

Expand $\dfrac{1}{\sqrt{(4+3x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

30. O/N 11/P32/Q3

The polynomial x^4+3x^3+ax+3 is denoted by $p(x)$. It is given that $p(x)$ is divisible by x^2-x+1.

(i) Find the value of a. [4]

(ii) When a has this value, find the real roots of the equation $p(x)=0$. [2]

31. O/N 11/P33/Q1

Expand $\dfrac{16}{(2+x)^2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

32. M/J 11/P32/Q8

(i) Express $\dfrac{5x-x^2}{(1+x)(2+x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{5x-x^2}{(1+x)(2+x^2)}$ in ascending powers of x, up to and including the term in x^3. [5]

33. M/J 11/P31/Q1

Expand $\sqrt[3]{(1-6x)}$ in ascending powers of x up to and including the term in x^3, simplifying the coefficients. [4]

34. O/N 10/P32/Q8, O/N 10/P31/Q8

Let $f(x)=\dfrac{3x}{(1+x)(1+2x^2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. [5]

35. O/N 10/P33/Q1

Expand $(1+2x)^{-3}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [3]

36. M/J 10/P33/Q9

(i) Express $\dfrac{4+5x-x^2}{(1-2x)(2+x)^2}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{4+5x-x^2}{(1-2x)(2+x)^2}$ in ascending powers of x, up to and including the term in x^2. [5]

37. O/N 09/P32/Q8

(i) Express $\dfrac{1+x}{(1-x)(2+x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{1+x}{(1-x)(2+x^2)}$ in ascending powers of x, up to and including the term in x^2. [5]

38. O/N 09/P31/Q8

(i) Express $\dfrac{5x+3}{(x+1)^2(3x+2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{5x+3}{(x+1)^2(3x+2)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [5]

39. M/J 09/P03/Q5

When $(1+2x)(1+ax)^{\frac{2}{3}}$, where a is a constant, is expanded in ascending powers of x, the coefficient of the term in x is zero.

(i) Find the value of a. [3]

(ii) When a has this value, find the term in x^3 in the expansion of $(1+2x)(1+ax)^{\frac{2}{3}}$, simplifying the coefficient. [4]

40. O/N 08/P03/Q2

Expand $(1+x)\sqrt{(1-2x)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

41. O/N 07/P03/Q9

(i) Express $\dfrac{2-x+8x^2}{(1-x)(1+2x)(2+x)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{2-x+8x^2}{(1-x)(1+2x)(2+x)}$ in ascending powers of x, up to and including the term in x^2. [5]

42. M/J 07/P03/Q1

Expand $(2+3x)^{-2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

43. O/N 06/P03/Q5

(i) Simplify $\left(\sqrt{(1+x)}+\sqrt{(1-x)}\right)\left(\sqrt{(1+x)}-\sqrt{(1-x)}\right)$, showing your working, and deduce that

$$\frac{1}{\sqrt{(1+x)}+\sqrt{(1-x)}}=\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{2x}.$$

(ii) Using this result, or otherwise, obtain the expansion of

$$\frac{1}{\sqrt{(1+x)}+\sqrt{(1-x)}}$$

in ascending powers of x, up to and including the term in x^2

44. M/J 06/P03/Q9

(i) Express $\dfrac{10}{(2-x)(1+x^2)}$ in partial fractions. [5]

(ii) Hence, given that $|x|<1$, obtain the expansion of $\dfrac{10}{(2-x)(1+x^2)}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [5]

45. O/N 05/P03/Q9

(i) Express $\dfrac{3x^2+x}{(x+2)(x^2+1)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{3x^2+x}{(x+2)(x^2+1)}$ in ascending powers of x, up to and including the term in x^3. [5]

46. M/J 05/P03/Q1

Expand $(1+4x)^{-\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

47. O/N 04/P03/Q1

Expand $\dfrac{1}{(2+x)^3}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

48. O/N 04/P03/Q3

The polynomial $2x^3 + ax^2 - 4$ is denoted by $p(x)$. It is given that $(x-2)$ is a factor of $p(x)$.

(i) Find the value of a . [2]

When a has this value,

(ii) factorise $p(x)$, [2]

(iii) solve the inequality $p(x) > 0$, justifying your answer. [2]

49. M/J 04/P03/Q9

Let $f(x) = \dfrac{x^2 + 7x - 6}{(x-1)(x-2)(x+1)}$.

(i) Express $f(x)$ in partial fractions. [4]

(ii) Show that, when x is sufficiently small for x^4 and higher powers to be neglected,

$$f(x) = -3 + 2x - \tfrac{3}{2}x^2 + \tfrac{11}{4}x^3.$$ [5]

50. O/N 03/P03/Q2

Expand $(2+x^2)^{-2}$ in ascending powers of x, up to and including the term in x^4, simplifying the coefficients. [4]

51. M/J 03/P03/Q6

Let $f(x) = \dfrac{9x^2 + 4}{(2x+1)(x-2)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that, when x is sufficiently small for x^3 and higher powers to be neglected,

$$f(x) = 1 - x + 5x^2.$$ [4]

52. O/N 02/P03/Q6

Let $f(x) = \dfrac{6+7x}{(2-x)(1+x^2)}$.

(i) Express $f(x)$ in partial fractions. [4]

(ii) Show that, when x is sufficiently small for x^4 and higher powers to be neglected,

$$f(x) = 3 + 5x - \tfrac{1}{2}x^2 - \tfrac{15}{4}x^3.$$ [5]

53. M/J 02/P03/Q2

Expand $(1-3x)^{-\frac{1}{3}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

Answers Section

1. M/J 18/P32/Q9

(i) Use a correct method to find a constant
Obtain one of the values $A=-3$, $B=1$, $C=2$
Obtain a second value
Obtain the third value **4**

(ii) Use a correct method to find the first two terms of the expansion of $(3-x)^{-1}$, $\left(1-\frac{1}{3}x\right)^{-1}$, $\left(2+x^2\right)^{-1}$ or $\left(1+\frac{1}{2}x^2\right)^{-1}$
Obtain correct unsimplified expansions up to the term in x^3 of each partial fraction
Multiply out their expansion, up to the terms in x^3, by $Bx+C$, where $BC \neq 0$
Obtain **final** answer $\frac{1}{6}x-\frac{11}{18}x^2-\frac{31}{108}x^3$, or equivalent **5**

2. M/J 18/P31/Q9

(i) State or imply the form $A+\frac{B}{x-1}+\frac{C}{3x+2}$
State or obtain $A=4$
Use a correct method to obtain a constant
Obtain one of $B=3$, $C=-1$
Obtain the other value **5**

(ii) Use correct method to find the first two terms of the expansion of $(x-1)^{-1}$ or $(3x+2)^{-1}$, or equivalent
Obtain correct unsimplified expansions up to the term in x^2 of each partial fraction
Add the value of A to the sum of the expansions
Obtain final answer $\frac{1}{2}-\frac{9}{4}x-\frac{33}{8}x^2$ **5**

3. M/J 18/P33/Q1

Obtain a correct unsimplified version of the x or x^2 term of the expansion of $(4-3x)^{\frac{1}{2}}$ or $\left(1-\frac{3}{4}x\right)^{\frac{1}{2}}$
State correct first term 2
Obtain the next two terms $\frac{3}{4}x+\frac{27}{64}x^2$ **4**

4. O/N 17/P32/Q8

(i) State or imply the form $\frac{A}{1-x}+\frac{B}{2x+3}+\frac{C}{(2x+3)^2}$
Use a relevant method to determine a constant
Obtain one of the values $A=1$, $B=-2$, $C=5$
Obtain a second value
Obtain the third value **5**
[Mark the form $\frac{A}{1-x}+\frac{Dx+E}{(2x+3)^2}$, where $A=1$, $D=-4$, $E=-1$, **B1M1A1A1A1** as above.]

(ii) Use a correct method to find the first two terms of the expansion of $(1-x)^{-1}$, $(1+\frac{2}{3}x)^{-1}$, $(2x+3)^{-1}$, $(1+\frac{2}{3}x)^{-2}$ or $(2x+3)^{-2}$
Obtain correct unsimplified expansions up to the term in x^2 of each partial fraction
Obtain final answer $\frac{8}{9}+\frac{19}{27}x+\frac{13}{9}x^2$, or equivalent 5

5. M/J 17/P32/Q8

(i) State or imply the form $\dfrac{A}{3x+2}+\dfrac{Bx+C}{x^2+5}$
Use a relevant method to determine a constant
Obtain one of the values $A=2$, $B=1$, $C=-3$
Obtain a second value
Obtain the third value 5

(ii) Use correct method to find the first two terms of the expansion of $(3x+2)^{-1}$, $(1+\frac{3}{2}x)^{-1}$, $(5+x^2)^{-1}$ or $(1+\frac{1}{5}x^2)^{-1}$
[Symbolic coefficients, e.g. $\begin{pmatrix}-1\\2\end{pmatrix}$ are not sufficient]
Obtain correct unsimplified expansions up to the term in x^2 of each partial fraction.
The FT is on A, B, C. from part (i)
Multiply out up to the term in x^2 by $Bx+C$, where $BC \neq 0$
Obtain **final answer** $\frac{2}{5}-\frac{13}{10}x+\frac{237}{100}x^2$, or equivalent 5

6. M/J 17/P31/Q2

EITHER:
State a correct unsimplified version of the x or x^2 or x^3 term in the expansion of $(1+6x)^{-\frac{1}{3}}$
State correct first two terms $1-2x$
Obtain term $8x^2$
Obtain term $-\frac{112}{3}x^3$ $\left(37\frac{1}{3}x^3\right)$ in final answer

OR:
Differentiate expression and evaluate $f(0)$ and $f'(0)$, where $f'(x)=k(1+6x)^{-\frac{4}{3}}$
Obtain correct first two terms $1-2x$
Obtain term $8x^2$
Obtain term $-\frac{112}{3}x^3$ in final answer 4

7. M/J 17/P33/Q2

EITHER:
State a correct unsimplified version of the x or x^2 term in the expansion of $\left(1+\frac{2}{3}x\right)^{-3}$ or $(3+2x)^{-3}$
[Symbolic binomial coefficients, e.g. $\begin{pmatrix}-3\\2\end{pmatrix}$, are not sufficient for M1]
State correct first term $\frac{1}{27}$
Obtain term $-\frac{2}{27}x$
Obtain term $\frac{8}{81}x^2$

OR:
Differentiate expression and evaluate $f(0)$ and $f'(0)$, where $f'(x)=k(3+2x)^{-4}$
State correct first term $\frac{1}{27}$
Obtain term $-\frac{2}{27}x$
Obtain term $\frac{8}{81}x^2$ 4

8. O/N 16/P32/Q2, O/N 16/P31/Q2

State correct unsimplified first two terms of the expansion of $(1+2x)^{-\frac{3}{2}}$, e.g. $1+(-\frac{3}{2})(2x)$
State correct unsimplified term in x^2, e.g. $(-\frac{3}{2})(-\frac{3}{2}-1)(2x)^2/2!$
Obtain sufficient terms of the product of $(2-x)$ and the expansion up to the term in x^2
Obtain final answer $2-7x+18x^2$ Do not ISW [4]

9. O/N 16/P33/Q8

(i) State or imply the form $\frac{A}{x+2}+\frac{Bx+C}{x^2+4}$
Use a correct method to determine a constant
Obtain one of $A=2$, $B=1$, $C=-1$
Obtain a second value
Obtain a third value [5]

(ii) Use correct method to find the first two terms of the expansion of $(x+2)^{-1}$, $(1+\frac{1}{2}x)^{-1}$, $(4+x^2)^{-1}$ or $(1+\frac{1}{4}x^2)^{-1}$
Obtain correct unsimplified expansions up to the term in x^2 of each partial fraction
Multiply out fully by $Bx+C$, where $BC\neq 0$
Obtain final answer $\frac{3}{4}-\frac{1}{4}x+\frac{5}{16}x^2$, or equivalent [5]
[Symbolic binomial coefficients, e.g. $\binom{-1}{1}$ are not sufficient for the M1. The f.t. is on A, B, C.]
[In the case of an attempt to expand $(3x^2+x+6)(x+2)^{-1}(x^2+4)^{-1}$, give M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for the final answer.]

10. M/J 16/P32/Q2

State a correct un-simplified version of the x or x^2 or x^3 term
State correct first two terms $1+x$
Obtain the next two terms $\frac{3}{2}x^2+\frac{5}{2}x^3$ [4]
[Symbolic binomial coefficients, e.g. $\binom{-\frac{1}{2}}{3}$ are not sufficient for the M mark.]

11. M/J 16/P31/Q8

(i) State or imply the form $\frac{A}{x+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$
Use a correct method to determine a constant
Obtain one of the values $A=1$, $B=3$, $C=12$
Obtain a second value
Obtain a third value [5]
[Mark the form $\frac{A}{x+1}+\frac{Dx+E}{(x-3)^2}$, where $A=1$, $D=3$, $E=3$, B1M1A1A1A1 as above.]

(ii) Use correct method to find the first two terms of the expansion of $(x+1)^{-1}$, $(x-3)^{-1}$, $(1-\frac{1}{3}x)^{-1}$, $(x-3)^{-2}$, or $(1-\frac{1}{3}x)^{-2}$
Obtain correct unsimplified expansions up to the term in x^2 of each partial fraction
Obtain final answer $\frac{4}{3}-\frac{4}{9}x+\frac{4}{3}x^2$, or equivalent [5]

12. M/J 16/P33/Q10

(i) State or imply the form $\dfrac{A}{x+3}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$

Use a correct method to determine a constant
Obtain one of the values $A=-3$, $B=1$, $C=2$
Obtain a second value
Obtain the third value

[Mark the form $\dfrac{A}{x+3}+\dfrac{Dx+E}{(x-1)^2}$, where $A=-3$, $D=1$, $E=1$, B1M1A1A1A1 as above.] [5]

(ii) Use a correct method to find the first two terms of the expansion of $(x+3)^{-1}$, $(1+\frac{1}{3}x)^{-1}$, $(x-1)^{-1}$, $(1-x)^{-1}$, $(x-1)^{-2}$, or $(1-x)^{-2}$

Obtain correct unsimplified expressions up to the term in x^2 of each partial fraction
Obtain final answer $\frac{10}{3}x+\frac{44}{9}x^2$, or equivalent [5]

13. O/N 15/P33/Q2

<u>Either</u> State correct unsimplified x^2 or x^3 term
Obtain $a=-9$
Obtain $b=45$

<u>Or</u> Use chain rule to differentiate twice to obtain form $k(1+9x)^{-\frac{5}{3}}$
Obtain $f''(x)=-18(1+9x)^{-\frac{5}{3}}$ and hence $a=-9$
Obtain $f'''(x)=270(1+9x)^{-\frac{8}{3}}$ and hence $b=45$ [3]

14. M/J 15/P32/Q8

(i) State or imply the form $\dfrac{A}{3-2x}+\dfrac{Bx+C}{x^2+4}$

Use a relevant method to determine a constant
Obtain one of the values $A=3$, $B=-1$, $C=-2$
Obtain a second value
Obtain the third value [5]

(ii) Use correct method to find the first two terms of the expansion of $(3-2x)^{-1}$, $(1-\dfrac{2}{3}x)^{-1}$, $(4+x^2)^{-1}$ or $(1+\dfrac{1}{4}x^2)^{-1}$

Obtain correct unsimplified expansions up to the term in x^2 of each partial fraction
Multiply out up to the term in x^2 by $Bx+C$, where $BC\neq 0$
Obtain final answer $\dfrac{1}{2}+\dfrac{5}{12}x+\dfrac{41}{72}x^2$, or equivalent [5]

[Symbolic coefficients, e.g. $\begin{pmatrix}-1\\2\end{pmatrix}$ are not sufficient for the first M1. The f.t. is on A, B, C.]

[In the case of an attempt to expand $(5x^2+x+6)(3-2x)^{-1}(x^2+4)^{-1}$, give M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for the final answer.]

15. M/J 15/P31/Q3

<u>Either</u> Obtain correct (unsimplified) version of x^2 or x^4 term in $(1-2x^2)^{-2}$
Obtain $1+4x^2$
Obtain $\ldots+12x^4$
Obtain correct (unsimplified) version of x^2 or x^4 term in $(1+6x^2)^{\frac{2}{3}}$
Obtain $1+4x^2-4x^4$
Combine expansions to obtain $k=16$ with no error seen

Or Obtain correct (unsimplified) version of x^2 or x^4 term in $(1+6x^2)^{\frac{2}{3}}$
Obtain $1+4x^2$
Obtain ... $-4x^4$
Obtain correct (unsimplified) version of x^2 or x^4 term in $(1-2x^2)^{-2}$
Obtain $1+4x^2+12x^4$
Combine expansions to obtain $k=16$ with no error seen [6]

16. O/N 14/P32/Q3

Substitute $x=-\frac{1}{3}$, equate result to zero or divide by $3x+1$ and equate the remainder to zero and obtain a correct equation, e.g. $-\frac{1}{27}a+\frac{1}{9}b-\frac{1}{3}+3=0$
Substitute $x=2$ and equate result to 21 or divide by $x-2$ and equate constant remainder to 21
Obtain a correct equation, e.g. $8a+4b+5=21$
Solve for a or for b
Obtain $a=12$ and $b=-20$ [5]

17. O/N 14/P32/Q9, O/N 14/P31/Q9

(i) State or imply the form $\frac{A}{1-x}+\frac{B}{2-x}+\frac{C}{(2-x)^2}$
Use a correct method to determine a constant
Obtain one of $A=2, B=-1, C=3$
Obtain a second value
Obtain a third value [5]
[The alternative form $\frac{A}{1-x}+\frac{Dx+E}{(2-x)^2}$, where $A=2, D=1, E=1$ is marked B1M1A1A1A1 as above.]

(ii) Use correct method to find the first two terms of the expansion of $(1-x)^{-1}, (2-x)^{-1}, (2-x)^{-2}, (1-\frac{1}{2}x)^{-1}$ or $(1-\frac{1}{2}x)^{-2}$
Obtain correct unsimplified expansions up to the term in x^2 of each partial fraction
Obtain final answer $\frac{9}{4}+\frac{5}{2}x+\frac{39}{16}x^2$, or equivalent [5]
[Symbolic binomial coefficients, e.g. $\begin{pmatrix}-1\\1\end{pmatrix}$ are not sufficient for M1. The ✓ is on A,B,C.]
[For the A,D,E form of partial fractions, give M1 A1✓ A1✓ for the expansions then, if $D\neq 0$, M1 for multiplying out fully and A1 for the final answer.]
[In the case of an attempt to expand $(x^2-8x+9)(1-x)^{-1}(2-x)^{-2}$, give M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for the final answer.]

18. M/J 14/P31/Q9

(i) Either State or imply partial fractions are of form $\frac{A}{3-x}+\frac{B}{1+2x}+\frac{C}{(1+2x)^2}$
Use any relevant method to obtain a constant
Obtain $A=1$
Obtain $B=\frac{3}{2}$
Obtain $C=-\frac{1}{2}$ [5]

Or State or imply partial fractions are of form $\frac{A}{3-x}+\frac{Dx+E}{(1+2x)^2}$

Use any relevant method to obtain a constant
Obtain $A=1$
Obtain $D=3$
Obtain $E=1$ [5]

(ii) Obtain the first two terms of one of the expansion of $(3-x)^{-1}, \left(1-\frac{1}{3}x\right)^{-1}$ $(1+2x)^{-1}$ and $(1+2x)^{-2}$

Obtain correct unsimplified expansion up to the term in x^2 of each partial fraction, following in each case the value of A, B, C

Obtain answer $\frac{4}{3}-\frac{8}{9}x+\frac{1}{27}x^2$ [5]

[If A, D, E approach used in part (i), give M1A1√A1√ for the expansions, M1 for multiplying out fully and A1 for final answer]

19. M/J 14/P33/Q2

State a correct unsimplified version of the x or x^2 or x^3 term
State correct first two terms $1-x$
Obtain the next two terms $2x^2-\frac{14}{3}x^3$ 4

[Symbolic binomial coefficients, e.g. $\begin{pmatrix}-\frac{1}{3}\\3\end{pmatrix}$ are not sufficient for the M mark.]

20. O/N 13/P32/Q7

(i) State or imply partial fractions are of the form $\frac{A}{x-2}+\frac{Bx+C}{x^2+3}$

Use a relevant method to determine a constant
Obtain one of the values $A=-1$, $B=3$, $C=-1$
Obtain a second value
Obtain the third value [5]

(ii) Use correct method to obtain the first two terms of the expansions of $(x-2)^{-1}$, $\left(1-\frac{1}{2}x\right)^{-1}$, $\left(x^2+3\right)^{-1}$ or $\left(1+\frac{1}{3}x^2\right)^{-1}$

Substitute correct unsimplified expansions up to the term in x^2 into each partial fraction
Multiply out fully by $Bx+C$, where $BC\neq 0$

Obtain final answer $\frac{1}{6}+\frac{5}{4}x+\frac{17}{72}x^2$, or equivalent [5]

[Symbolic binomial coefficients, e.g. $\begin{pmatrix}-1\\1\end{pmatrix}$ are not sufficient for the M1. The f.t. is on A, B, C.]

[In the case of an attempt to expand $\left(2x^2-7x-1\right)\left(x-2\right)^{-1}\left(x^2+3\right)^{-1}$, give M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for the final answer.]
[If B or C omitted from the form of partial fractions, give B0M1A0A0A0 in **(i)**; M1A1√A1√ in **(ii)**]

21. O/N 13/P33/Q8

(i) Either State or imply form $\frac{A}{1+x}+\frac{B}{(1+x)^2}+\frac{C}{2-3x}$

Use any relevant method to find at least one constant
Obtain $A=-1$
Obtain $B=3$
Obtain $C=4$

Or State or imply form $\dfrac{A}{1+x}+\dfrac{Bx}{(1+x)^2}+\dfrac{C}{2-3x}$
Use any relevant method to find at least one constant
Obtain $A=2$
Obtain $B=-3$
Obtain $C=4$

Or State or imply form $\dfrac{Dx+E}{(1+x)^2}+\dfrac{F}{2-3x}$
Use any relevant method to find at least one constant
Obtain $D=-1$
Obtain $E=2$
Obtain $F=4$ [5]

(ii) Either Use correct method to find first two terms of expansion of $(1+x)^{-1}$ or $(1+x)^{-2}$ or $(2-3x)^{-1}$ or $\left(1-\dfrac{3}{2}x\right)^{-1}$
Obtain correct unsimplified expansion of first partial fraction up to x^2 term
Obtain correct unsimplified expansion of second partial fraction up to x^2 term
Obtain correct unsimplified expansion of third partial fraction up to x^2 term
Obtain final answer $4-2x+\dfrac{25}{2}x^2$

Or 1 Use correct method to find first two terms of expansion of $(1+x)^{-2}$ or $(2-3x)^{-1}$ or $\left(1-\dfrac{3}{2}x\right)^{-1}$
Obtain correct unsimplified expansion of first partial fraction up to x^2 term
Obtain correct unsimplified expansion of second partial fraction up to x^2 term
Expand and obtain sufficient terms to obtain three terms
Obtain final answer $4-2x+\dfrac{25}{2}x^2$

Or 2 (expanding original expression)
Use correct method to find first two terms of expansion of $(1+x)^{-2}$ or $(2-3x)^{-1}$ or $\left(1-\dfrac{3}{2}x\right)^{-1}$
Obtain correct expansion $1-2x+3x^2$ or unsimplified equivalent
Obtain correct expansion $\dfrac{1}{2}\left(1+\dfrac{3}{2}x+\dfrac{9}{4}x^2\right)$ or unsimplified equivalent
Expand and obtain sufficient terms to obtain three terms
Obtain final answer $4-2x+\dfrac{25}{2}x^2$

Or 3 (McLaurin expansion)
Obtain first derivative $f'(x)=(1+x)^{-2}-6(1+x)^{-3}+12(2-3x)^{-2}$
Obtain $f'(0)=1-6+3$ or equivalent
Obtain $f''(0)=-2+18+9$ or equivalent
Use correct form for McLaurin expansion
Obtain final answer $4-2x+\dfrac{25}{2}x^2$ [5]

22. M/J 13/P32/Q8(i)

(i) Use any relevant method to determine a constant
Obtain one of the values $A = 1, B = -2, C = 4$
Obtain a second value
Obtain the third value [4]
[If A and C are found by the cover up rule, give B1 + B1 then M1A1 for finding B. If only one is found by the rule, give B1M1A1A1.]

23. M/J 13/P31/Q1

Carry out division or equivalent at least as far as two terms of quotient
Obtain quotient $2x-4$
Obtain remainder 8 [3]

24. M/J 13/P31/Q2

Obtain $1-x$ as first two terms of $(1+2x)^{-\frac{1}{2}}$
Obtain $+\frac{3}{2}x^2$ or unsimplified equivalent as third term of $(1+2x)^{-\frac{1}{2}}$
Multiply $1+3x$ by attempt at $(1+2x)^{-\frac{1}{2}}$, obtaining sufficient terms
Obtain final answer $1+2x-\frac{3}{2}x^2$ [4]

25. O/N 12/P32/Q4, O/N 12/P31/Q4

(i) Obtain correct unsimplified terms in x and x^3
Equate coefficients and solve for a
Obtain final answer $a=\frac{1}{\sqrt{2}}$, or exact equivalent [4]

(ii) Use correct method and value of a to find the first two terms of the expansion $(1 + ax)^{-2}$
Obtain $1-\sqrt{2}x$, or equivalent
Obtain term $\frac{3}{2}x^2$ [3]
[Symbolic coefficients, e.g. $\binom{-2}{1}a$, are not sufficient for the first B marks]
[The f.t. is solely on the value of a.]

26. O/N 12/P33/Q9

(i) State or imply form $\frac{A}{3-x}+\frac{Bx+C}{1+x^2}$
Use relevant method to determine a constant
Obtain $A = 6$
Obtain $B = -2$
Obtain $C = 1$ [5]

(ii) Either Use correct method to obtain first two terms of expansion of $(3-x)^{-1}$ or $\left(1-\frac{1}{3}x\right)^{-1}$ or $\left(1+x^2\right)^{-1}$
Obtain $\frac{A}{3}\left(1+\frac{1}{3}x+\frac{1}{9}x^2+\frac{1}{27}x^3\right)$
Obtain $(Bx + C)(1-x^2)$
Obtain sufficient terms of the product $(Bx + C)(1-x^2)$, $B, C \neq 0$ and add the two expansions
Obtain final answer $3-\frac{4}{3}x-\frac{7}{9}x^2+\frac{56}{27}x^3$

Or Use correct method to obtain first two terms of expansion of $(3-x)^{-1}$ or $\left(1-\frac{1}{3}x\right)^{-1}$ or $\left(1+x^2\right)^{-1}$

Obtain $\frac{1}{3}\left(1+\frac{1}{3}x+\frac{1}{9}x^2+\frac{1}{27}x^3\right)$

Obtain $(1-x^2)$

Obtain sufficient terms of the product of the three factors

Obtain final answer $3-\frac{4}{3}x-\frac{7}{9}x^2+\frac{56}{27}x^3$ [5]

27. M/J 12/P32/Q3

EITHER: State a correct unsimplified term in x or x^2 of $(1-x)^{\frac{1}{2}}$ or $(1+x)^{-\frac{1}{2}}$

State correct unsimplified expansion of $(1-x)^{\frac{1}{2}}$ up to the term in x^2

State correct unsimplified expansion of $(1+x)^{-\frac{1}{2}}$ up to the term in x^2

Obtain sufficient terms of the product of the expansions of $(1-x)^{\frac{1}{2}}$ and $(1+x)^{-\frac{1}{2}}$

Obtain final answer $1-x+\frac{1}{2}x^2$

*OR*1: State that the given expression equals $(1-x)(1-x^2)^{-\frac{1}{2}}$ and state that the first term of the expansion of $(1-x^2)^{-\frac{1}{2}}$ is 1 B1

State correct unsimplified term in x^2 of $(1-x^2)^{-\frac{1}{2}}$

State correct unsimplified expansion of $(1-x^2)^{-\frac{1}{2}}$ up to the term in x^2

Obtain sufficient terms of the product of $(1-x)$ and the expansion M1

Obtain final answer $1-x+\frac{1}{2}x^2$

*OR*2: State correct unsimplified expansion of $(1+x)^{\frac{1}{2}}$ up to the term in x^2

Multiply expansion by $(1-x)$ and obtain $1-2x+2x^2$

Carry out correct method to obtain one non-constant term of the expansion of $\left(1-2x+2x^2\right)^{\frac{1}{2}}$

Obtain a correct unsimplified expansion with sufficient terms A1

Obtain final answer $1-x+\frac{1}{2}x^2$ [5]

[Treat $(1+x)^{-1}(1-x^2)^{\frac{1}{2}}$ by the *EITHER* scheme.]

[Symbolic coefficients, e.g. $\begin{pmatrix}\frac{1}{2}\\2\end{pmatrix}$, are not sufficient for the B marks.]

28. M/J 12/P31/Q2

(i) Either Obtain correct (unsimplified) version of x or x^2 term from $(1-4x)^{\frac{1}{2}}$

Obtain $1+2x$

Obtain $+6x^2$

Or Differentiate and evaluate f(0) and f'(0) where $f(x)=k(1-4x)^{-\frac{1}{2}}$

Obtain $1+2x$

Obtain $+6x^2$

(ii) Combine both x^2 terms from product of $1+2x$ and answer from part **(i)**

Obtain 5 [2]

29. M/J 12/P33/Q1

EITHER: Obtain a correct unsimplified version of the x or x^2 term of the expansion of $(4+3x)^{-\frac{1}{2}}$ or $(1+\frac{3}{4}x)^{-\frac{1}{2}}$

State correct first term $\frac{1}{2}$

Obtain the next two terms $-\frac{3}{16}x+\frac{27}{256}x^2$

OR: Differentiate and evaluate f(0) and f ′(0), where $f'(x)=k(4+3x)^{-\frac{3}{2}}$

State correct first term $\frac{1}{2}$

Obtain the next two terms $-\frac{3}{16}x+\frac{27}{256}x^2$ [4]

[Symbolic coefficients, e.g. $\begin{pmatrix}-\frac{1}{2}\\ 2\end{pmatrix}$ are not sufficient for the M or B mark.]

30. O/N 11/P32/Q3

(i) *EITHER*: Attempt division by x^2-x+1 reaching a partial quotient of x^2+kx
Obtain quotient x^2+4x+3
Equate remainder of form lx to zero and solve for a, or equivalent
Obtain answer $a=1$

OR: Substitute a complex zero of x^2-x+1 in p(x) and equate to zero
Obtain a correct equation in a in any unsimplified form
Expand terms, use $i^2=-1$ and solve for a
Obtain answer $a=1$ [4]

[SR: The first M1 is earned if inspection reaches an unknown factor x^2+Bx+C and an equation in B and/or C, or an unknown factor Ax^2+Bx+3 and an equation in A and/or B. The second M1 is only earned if use of the equation $a=B-C$ is seen or implied.]

(ii) State answer, e.g. $x=-3$
State answer, e.g. $x=-1$ and no others [2]

31. O/N 11/P33/Q1

Either
Obtain correct unsimplified version of x or x^2 term in expansion of

$(2+x)^{-2}$ or $(1+\frac{1}{2}x)^{-2}$

Correct first term 4 from correct work
Obtain $-4x$
Obtain $+3x^2$

Or
Differentiate and evaluate f(0) and f′ (0) where $f'(x)=k(2+x)^{-3}$
State correct first term 4
Obtain $-4x$
Obtain $+3x^2$ [4]

32. M/J 11/P32/Q8

(i) State or imply partial fractions are of the form $\frac{A}{1+x}+\frac{Bx+C}{2+x^2}$

Use a relevant method to determine a constant
Obtain one of the values $A=-2$, $B=1$, $C=4$
Obtain a second value
Obtain the third value [5]

(ii) Use correct method to obtain the first two terms of the expansion of $(1+x)^{-1}$, $\left(1+\frac{1}{2}x^2\right)^{-1}$ or $\left(2+x^2\right)^{-1}$ in ascending powers of x

Obtain correct unsimplified expansion up to the term in x^3 of each partial fraction
Multiply out fully by $Bx + C$, where $BC \neq 0$

Obtain final answer $\frac{5}{2}x - 3x^2 + \frac{7}{4}x^3$, or equivalent [5]

[Symbolic binomial coefficients, e.g. $\begin{pmatrix}-1\\1\end{pmatrix}$, are not sufficient for the first M1. The f.t. is on A, B, C.]
[If B or C omitted from the form of fractions, give B0M1A0A0A0 in **(i)**; M1A1√A1√ in **(ii)**, max 4/10.]
[In the case of an attempt to expand $(5x - x^2)(1 + x)^{-1}(2 + x^2)^{-1}$, give M1A1A1 for the expansions, M1 for the multiplying out fully, and A1 for the final answer.]
[Allow use of Maclaurin, giving M1A1√A1√ for differentiating and obtaining f(0) = 0 and $f'(0) = \frac{5}{2}$, A1√ for $f''(0) = -6$, and A1 for $f'''(0) = \frac{21}{2}$ and the final answer (the f.t. is on A, B, C if used).]
[For the identity $5x - x^2 \equiv \left(2 + 2x + x^2 + x^3\right)\left(a + bx + cx^2 + dx^3\right)$ give M1A1; then M1A1 for using a relevant method to obtain two of $a = 0$, $b = \frac{5}{2}$, $c = -3$ and $d = \frac{7}{4}$; then A1 for the final answer in series form.]

33. M/J 11/P31/Q1

Either: Obtain $1+\frac{1}{3}kx$, where $k = \pm 6$ or ± 1
Obtain $1 - 2x$
Obtain $-4x^2$
Obtain $-\frac{40}{3}x^3$ or equivalent

Or: Differentiate expression to obtain form $k(1-6x)^{-\frac{2}{3}}$ and evaluate f(0) and f′(0)
Obtain $f'(x) = -2(1-6x)^{-\frac{2}{3}}$ and hence the correct first two terms $1 - 2x$
Obtain $f''(x) = -8(1-6x)^{-\frac{5}{3}}$ and hence $-4x^2$
Obtain $f'''(x) = -80(1-6x)^{-\frac{8}{3}}$ and hence $-\frac{40}{3}x^3$ or equivalent [4]

34. O/N 10/P32/Q8, O/N 10/P31/Q8

(i) State or imply the form $\frac{A}{1+x} + \frac{Bx+C}{1+2x^2}$
Use any relevant method to evaluate a constant
Obtain one of $A = -1$, $B = 2$, $C = 1$
Obtain a second value
Obtain the third value [5]

(ii) Use correct method to obtain the first two terms of the expansion of $(1+x)^{-1}$ or $\left(1+2x^2\right)^{-1}$
Obtain correct expansion of each partial fraction as far as necessary
Multiply out fully by $Bx + C$, where BC Þ 0
Obtain answer $3x - 3x^2 - 3x^3$ [5]

[Symbolic binomial coefficients, e.g., $\begin{pmatrix}-1\\1\end{pmatrix}$ are not sufficient for the first M1. The f.t. is on A, B, C.]
[If B or C omitted from the form of fractions, give B0M1A0A0A0 in **(i)**; M1A1√A1√ in **(ii)**, max 4/10.]
[If a constant D is added to the correct form, give M1A1A1A1 and B1 if and only if $D = 0$ is stated.]
[If an extra term $D/(1+2x^2)$ is added, give B1M1A1A1, and A1 if $C + D = 1$ is resolved to $1/(1+2x^2)$.]
[In the case of an attempt to expand $3x(1+x)^{-1}(1+2x^2)^{-1}$, give M1A1A1 for the expansions up to the term in x^2, M1 for multiplying out fully, and A1 for the final answer.]
[For the identity $3x \equiv (1 + x + 2x^2 + 2x^3)(a + bx + cx^2 + dx^3)$ give M1A1; then M1A1 for using a relevant method to find two of $a = 0$, $b = 3$, $c = -3$ and $d = -3$; and then A1 for the final answer in series form.]

35. O/N 10/P33/Q1

Obtain $1 - 6x$
State correct unsimplified x^2 term. Binomial coefficients must be expanded.
Obtain $\ldots + 24x^2$ [3]

36. M/J 10/P33/Q9

(i) State or imply partial fractions of the form $\frac{A}{1-2x}+\frac{B}{2+x}+\frac{C}{(2+x)^2}$

Use any relevant method to determine a constant
Obtain one of the values $A = 1$, $B = 1$, $C = -2$
Obtain a second value
Obtain the third value [5]

[The form $\frac{A}{1-2x}+\frac{Dx+E}{(2+x)^2}$, where $A = 1$, $D = 1$, $E = 0$, is acceptable scoring B1M1A1A1A1 as above.]

(ii) Use correct method to obtain the first two terms of the expansion of $(1-2x)^{-1}$, $(2+x)^{-1}$, $(2+x)^{-2}$, $(1+\frac{1}{2}x)^{-1}$, or $(1+\frac{1}{2}x)^{-2}$
Obtain correct unsimplified expansions up to the term in x^2 of each partial fraction A1√ + A1√ + A1√
Obtain answer $1+\frac{9}{4}x+\frac{15}{4}x^2$, or equivalent [5]

[Symbolic binomial coefficients, e.g. $\begin{pmatrix}-1\\1\end{pmatrix}$, are not sufficient for the M1. The f.t. is on A, B, C.]

[For the A, D, E form of partial fractions, give M1A1√A1√ for the expansions then, if $D \neq 0$, M1 for multiplying out fully and A1 for the final answer.]

[In the case of an attempt to expand $(4+5x-x^2)(1-2x)^{-1}(2+x)^{-2}$, give M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for the final answer.]
[SR: If B or C omitted from the form of fractions, give B0M1A0A0A0 in (i); M1A1√A1√ in (ii).]
[SR: If D or E omitted from the form of fractions, give B0M1A0A0A0 in (i); M1A1√A1√ in (ii).]

37. O/N 09/P32/Q8

(i) State or imply partial fractions are of the form $\frac{A}{1-x}+\frac{Bx+C}{2+x^2}$

Use a relevant method to determine a constant
Obtain $A=\frac{2}{3}$, $B=\frac{2}{3}$ and $C=\frac{1}{3}$ [5]

(ii) Use correct method to find first two terms of the expansion of $(1-x)^{-1}$, $(2+x^2)^{-1}$ or $(1+\frac{1}{2}x^2)^{-1}$
Obtain complete unsimplified expansions up to x^2 of each partial fraction e.g. $\frac{2}{3}(1+x+x^2)$ and $\frac{1}{2}(\frac{2}{3}x-\frac{1}{3})(1-\frac{1}{2}x^2)$
Carry out multiplication of $(2+x^2)^{-1}$ by $(\frac{2}{3}x-\frac{1}{3})$, or equivalent, provided $BC \neq 0$
Obtain answer $\frac{1}{2}+x+\frac{3}{4}x^2$ [5]
[Symbolic binomial coefficients are not sufficient for the first M1. The f.t. is on A, B, C.]
[If B or C omitted from the form of fractions, give B0M1A0A0A0 in **(i)**; M1A1√A1√ in **(ii)**, max 4/10]
[In the case of an attempt to expand $(1+x)(1-x)^{-1}(2+x^2)^{-1}$, give M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for the final answer.]
[Allow Maclaurin, giving M1A1√A1√ for differentiating and obtaining $f(0)=\frac{1}{2}$ and $f'(0)=1$, A1√ for $f''(0)=\frac{3}{2}$, and A1 for the final answer (the f.t. is on A, B, C if used).]

38. O/N 09/P31/Q8

(i) State or imply partial fractions are of the form $\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{3x+2}$
Use any relevant method to obtain a constant
Obtain one of the values $A=1$, $B=2$, $C=-3$
Obtain a second value
Obtain the third value [5]

(ii) Use correct method to obtain the first two terms of the expansion of $(x+1)^{-1}$, $(x+1)^{-2}$, $(3x+2)^{-1}$ or $(1+\frac{3}{2}x)^{-1}$
Obtain correct unsimplified expansion up to the term in x^2 of each partial fraction
Obtain answer $\frac{3}{2}-\frac{11}{4}x+\frac{29}{8}x^2$, or equivalent [5]
[Symbolic binomial coefficients, e.g. $\begin{pmatrix}-1\\1\end{pmatrix}$, are not sufficient for the first M1. The f.t. is on A, B, C.]
[The form $\frac{Dx+E}{(x+1)^2}+\frac{C}{3x+2}$, where $D=1$, $E=3$, $C=-3$, is acceptable. In part **(i)** give B1M1A1A1A1.
In part **(ii)** give M1A1√A1√ for the expansions, and, if $DE \neq 0$, M1 for multiplying out fully and A1 for the final answer.]
[If B or C omitted from the form of fractions, give B0M1A0A0A0 in **(i)**; M1A1√A1√ in **(ii)**, max 4/10]
[If D or E omitted from the form of fractions, give B0M1A0A0A0 in **(i)**; M1A1√A1√ in **(ii)**, max 4/10]
[In the case of an attempt to expand $(5x+3)(x+1)^{-2}(3x+2)^{-1}$, give M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for the final answer.]
[Allow use of Maclaurin, giving M1A1√A1√ for differentiating and obtaining $f(0)=\frac{3}{2}$ and $f'(0)=-\frac{11}{4}$, A1√ for $f''(0)=\frac{29}{4}$, and A1 for the final answer (the f.t. is on A, B, C if used).]

39. M/J 09/P03/Q5

(i) State correct first two terms of the expansion of $(1+ax)^{\frac{2}{3}}$, i.e. $1+\frac{2}{3}ax$
Form an expression for the coefficient of x in the expansion of $(1+2x)(1+ax)^{\frac{2}{3}}$ and equate it to zero
Obtain $a=-3$ 3

(ii) Obtain correct unsimplified terms in x^2 and x^3 in the expansion of $(1-3x)^{\frac{2}{3}}$ or $(1+ax)^{\frac{2}{3}}$

Carry out multiplication by $1 + 2x$ obtaining two terms in x^3

Obtain final answer $-\frac{10}{3}x^3$, or equivalent 4

[Symbolic binomial coefficients, e.g. $\begin{pmatrix}\frac{2}{3}\\1\end{pmatrix}$, are not acceptable for the B marks in **(i)** or **(ii)**]

40. O/N 08/P03/Q2

EITHER: State correct unsimplified first two terms of the expansion of $\sqrt{(1-2x)}$, e.g. $1+\frac{1}{2}(-2x)$

State correct unsimplified term in x^2, e.g. $\frac{1}{2}.(\frac{1}{2}-1).(-2x)^2/2!$

Obtain sufficient terms of the product of $(1+x)$ and the expansion up to the term in x^2 of $\sqrt{(1-2x)}$

Obtain final answer $1-\frac{3}{2}x^2$

[The B marks are not earned by versions with symbolic binomial coefficients such as $\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}$.]

[SR: An attempt to rewrite $(1+x)\sqrt{(1-2x)}$ as $\sqrt{(1-3x^2)}$ earns M1 A1 and the subsequent expansion $1-\frac{3}{2}x^2$ gets M1 A1.]

OR: Differentiate expression and evaluate f(0) and f ′(0), having used the product rule

Obtain f(0) = 1 and f ′(0) = 0 correctly

Obtain f ″(0) = −3 correctly

Obtain final answer $1-\frac{3}{2}x^2$, with no errors seen [4]

41. O/N 07/P03/Q9

(i) State or imply the form $\frac{A}{1-x}+\frac{B}{1+2x}+\frac{C}{2+x}$

Use any relevant method to determine a constant

Obtain $A = 1$, $B = 2$ and $C = -4$ [5]

(ii) Use correct method to obtain the first two terms of the expansion of $(1-x)^{-1}$, $(1+2x)^{-1}$, $(2+x)^{-1}$, or $(1+\frac{1}{2}x)^{-1}$

Obtain complete unsimplified expansions up to x^2 of each partial fraction

Combine expansions and obtain answer $1-2x+\frac{17}{2}x^2$ [5]

[Binomial coefficients such as $\begin{pmatrix}-1\\2\end{pmatrix}$ are not sufficient for the M1. The f.t. is on *A*, *B*, *C*.]

[Apply this scheme to attempts to expand $(2-x+8x^2)(1-x)^{-1}(1+2x)^{-1}(2+x)^{-1}$, giving M1A1A1A1 for the expansions, and A1 for the final answer.]

[Allow Maclaurin, giving M1A1√A1√ for f(0) = 1 and f ′(0) = −2, A1√ for f ″(0) = 17 and A1 for the final answer (f.t. is on *A*, *B*, *C*).]

42. M/J 07/P03/Q1

EITHER: Obtain correct unsimplified version of the x or x^2 term in the expansion of $(2+3x)^{-2}$ or $(1+\frac{3}{2}x)^{-2}$

State correct first term $\frac{1}{4}$

Obtain the next two terms $-\frac{3}{4}x+\frac{27}{16}x^2$

[The M mark is not earned by versions with symbolic binomial coefficient such as $\begin{pmatrix}-2\\1\end{pmatrix}$.]

[The M mark is earned if division of 1 by the expansion of $(2+3x)^2$, with a correct unsimplified x or x^2 term, reaches a partial quotient of $a+bx$.]

[Accept exact decimal equivalents of fractions.]
[SR: Answer given as $\frac{1}{4}(1-3x+\frac{27}{4}x^2)$ can earn B1M1A1 (if $\frac{1}{4}$ seen but then omitted, give M1A1).]
[SR: Solutions involving $k(1+\frac{3}{2}x)^{-2}$, where $k = 2, 4$ or $\frac{1}{2}$, can earn M1 and A1√ for correctly Simplifying both the terms in x and x^2 .]

OR: Differentiate expression and evaluate f(0) and f′(0), where $f'(x) = k(2+3x)^{-3}$
State correct first term $\frac{1}{4}$
Obtain the next two terms $-\frac{3}{4}x+\frac{27}{16}x^2$ 4

43. O/N 06/P03/Q5

(i) Simplify product and obtain $(1+x)-(1-x)$
Complete the proof of the given result with no errors seen 2

(ii) Use correct method to obtain the first two terms of the expansion of $\sqrt{1+x}$ or $\sqrt{1-x}$
EITHER: Obtain any correct unsimplified expansion of the numerator of the RHS of the identity up to the terms in x^3
Obtain final answer with constant term $\frac{1}{2}$
Obtain term $\frac{1}{16}x^2$ and no term in x
OR: Obtain any correct unsimplified expansion of the denominator of the LHS of the identity up to the terms in x^2
Obtain final answer with constant term $\frac{1}{2}$
Obtain term $\frac{1}{16}x^2$ and no term in x 4
[Symbolic binomial coefficients are not sufficient for the M1. Allow two correct separate expansions to earn the first A1 if the context is clear and appropriate.]
[Allow the use of Maclaurin, giving M1A1 for $f(0) = \frac{1}{2}$ and $f'(0) = 0$, A1 for $f'(0) = \frac{1}{8}$, and A1 for obtaining the correct final answer.]

44. M/J 06/P03/Q9

(i) State or imply partial fractions are of the form $\frac{A}{2-x}+\frac{Bx+C}{1+x^2}$
Use any relevant method to obtain a constant
Obtain one of the values $A = 2, B = 2, C = 4$
Obtain a second value
Obtain the third value 5

(ii) Use correct method to obtain the first two terms of the expansion of $(2-x)^{-1}$ or $(1-\frac{1}{2}x)^{-1}$ or $(1+x^2)^{-1}$
Obtain any correct unsimplified expansion of the partial fractions up to the terms in x^3, e.g. $(2x+4)(1+(-1)x^2)$ (deduct A1 for each incorrect expansion)
Carry out multiplication of expansion of $(1+x^2)^{-1}$ by $(2x+4)$
Obtain answer $5+\frac{3}{2}x-\frac{15}{4}x^2-\frac{15}{8}x^3$ 5
[Binomial coefficients involving -1, e.g. $\begin{pmatrix}-1\\1\end{pmatrix}$, are not sufficient for the M1 mark. The f.t. is on A,B,C]
[In the case of an attempt to expand $10(2-x)^{-1}(1+x^2)^{-1}$, give M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for the final answer.]
[Allow the use of Maclaurin, giving M1A1√ for $f(0) = 5$ and $f'(0) = \frac{5}{2}$, A1√ for $f''(0) = -\frac{15}{2}$, A1√ for $f'''(0) = -\frac{45}{4}$, and A1 for obtaining the correct final answer (f.t. is on A,B,C if used).]

45. O/N 05/P03/Q9

(i) State or imply partial fractions are of the form $\frac{A}{x+2}+\frac{Bx+C}{x^2+1}$
Use any relevant method to obtain a constant
Obtain $A=2$
Obtain $B=1$
Obtain $C=-1$ [5]

(ii) Use correct method to obtain the first two terms of the expansion of $(2+x)^{-1}$, or $(1+\frac{1}{2}x)^{-1}$, or $(1+x^2)^{-1}$

Obtain complete unsimplified expansions of the fractions, e.g. $2.\frac{1}{2}(1-\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^3)$; $(x-1)(1-x^2)$

Carry out multiplication of expansion of $(1+x^2)^{-1}$ by $(x-1)$

Obtain answer $\frac{1}{2}x+\frac{5}{4}x^2-\frac{9}{8}x^3$ [5]

[Binomial coefficients involving −1, such as $\begin{pmatrix}-1\\1\end{pmatrix}$, are not sufficient for the first M1.]

[f.t. is on A, B, C.]

[Apply this scheme to attempts to expand $(3x^2+x)(x+2)^{-1}(1+x^2)^{-1}$, giving M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for the final answer.]

46. M/J 05/P03/Q1

EITHER: Obtain correct unsimplified version of the x or x^2 *or* x^3 term
State correct first two terms $1-2x$
Obtain next two terms $6x^2-20x^3$
[The M mark is not earned by versions with unexpanded binomial coefficients, e.g. $\begin{pmatrix}-\frac{1}{2}\\2\end{pmatrix}$.]

OR: Differentiate expression and evaluate f(0) and f′(0), where $f'(x)=k(1+4x)^{-\frac{3}{2}}$
State correct first two terms $1-2x$
Obtain next two terms $6x^2-20x^3$ 4

47. O/N 04/P03/Q1

EITHER: Obtain correct unsimplified version of the x or x^2 term in the expansion of $(2+x)^{-3}$ or $\left(1+\frac{1}{2}x\right)^{-3}$

State correct first term $\frac{1}{8}$

Obtain next two terms $-\frac{3}{16}x+\frac{3}{16}x^2$

[The M mark is not earned by versions with unexpanded binomial coefficients such as $\begin{pmatrix}-3\\1\end{pmatrix}$.]

[Accept exact decimal equivalents of fractions.]

[SR: Answers given as $\frac{1}{8}\left(1-\frac{3}{2}x+\frac{3}{2}x^2\right)$ can earn M1B1A1.]

[SR: Solutions involving $k\left(1+\frac{1}{2}x\right)^{-3}$, where k = 2, 8 or $\frac{1}{2}$, can earn M1 and A1√ for correctly simplifying both the terms in x and x^2.]

OR: Differentiate expression and evaluate f(0) and f′(0), where $f'(x) = k(2+x)^{-4}$

State correct first term $\frac{1}{8}$

Obtain next two terms $-\frac{3}{16}x+\frac{3}{16}x^2$ **4**

[Accept exact decimal equivalents of fractions.]

48. O/N 04/P03/Q3

(i) Substitute 2 for x and equate to zero, or divide by $x-2$ and equate remainder to zero

Obtain answer $a=-3$ **2**

(ii) Attempt to find quadratic factor by division or inspection

State quadratic factor $2x^2+x+2$ **2**

[The M1 is earned if division reaches a partial quotient of $2x^2+kx$, or if inspection has an unknown factor of $2x^2+bx+c$ and an equation in b and/or c, or if two coefficients with the correct moduli are stated without working.]

(iii) State answer $x>2$ (and nothing else)

Make a correct justification e.g. $2x^2+x+2$ (has no zeros and) is always positive **2**

[SR: The answer $x \geq 2$ gets B0, but in this case allow the second B mark if the remaining work is correct.]

49. M/J 04/P03/Q9

(i) State or imply $f(x) \equiv \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x+1}$

EITHER: Use any relevant method to obtain a constant

Obtain one of the values: $A=-1$, $B=4$ and $C=-2$

Obtain the remaining two values

OR: Obtain one value by inspection

State a second value

State the third value **4**

[Apply the same scheme to the form $\frac{A}{x-2}+\frac{Bx+C}{x^2-1}$ which has $A=4$, $B=-3$ and $C=1$.]

(ii) Use correct method to obtain the first two terms of the expansion of $(x-1)^{-1}$ or $(x-2)^{-1}$ or $(x+1)^{-1}$

Obtain any correct unsimplified expansion of the partial fractions up to the terms in x^3 (deduct A1 for each incorrect expansion)

Obtain the given answer correctly **5**

[Binomial coefficients involving −1, e.g. $\begin{pmatrix}-1\\1\end{pmatrix}$, are not sufficient for the M1 mark. The f.t. is on A, B, C.]

[Apply a similar scheme to the alternative form of fractions in (i), awarding M1 A1√A1√ for the expansions, M1(dep*) for multiplying by $Bx+C$, and A1 for obtaining the given answer correctly.]

[In the case of an attempt to expand $(x^2+7x-6)(x-1)^{-1}(x-2)^{-1}(x+1)^{-1}$, give M1A1A1A1 for the expansions and A1 for multiplying out and obtaining the given answer correctly.]

[Allow attempts to multiply out $(x-1)(x-2)(x+1)(-3+2x-\frac{3}{2}x^2+\frac{11}{4}x^3)$, giving B1 for reduction to a product of two expressions correct up to their terms in x^3, M1 for attempting to multiply out at least as far as terms in x^2, A1 for a correct expansion up to terms in x^3, and A1 for correctly obtaining the answer x^2+7x-6 and also showing there is no term in x^3.]

[Allow the use of Maclaurin, giving M1A1√ for f(0) = −3 and f′(0) = 2, A1√ for f″(0) = −3, A1√ for f‴(0) = $\frac{33}{2}$, and A1 for obtaining the given answer correctly (f.t. is on A, B, C if used).]

50. O/N 03/P03/Q2

EITHER: Obtain correct unsimplified version of the x^2 or x^4 term of the expansion of $(1+\frac{1}{2}x^2)^{-2}$ or $(2+x^2)^{-2}$

State correct first term $\frac{1}{4}$

Obtain next two terms $-\frac{1}{4}x^2+\frac{3}{16}x^4$

[The M mark is not earned by versions with unexpanded binomial coefficients such as $\begin{pmatrix}-2\\1\end{pmatrix}$.]

[SR: Answers given as $\frac{1}{4}(1-x^2+\frac{3}{4}x^4)$ earn M1B1A1.]

[SR: Solutions involving $k(1+\frac{1}{2}x^2)^{-2}$, where $k=2$, 4 or $\frac{1}{2}$ can earn M1 and A1 for a correct simplified term in x^2 or x^4.]

OR: Differentiate expression and evaluate f(0) and f ′(0), where $f'(x)=kx(2+x^2)^{-3}$

State correct first term $\frac{1}{4}$

Obtain next two terms $-\frac{1}{4}x^2+\frac{3}{16}x^4$

[Allow exact decimal equivalents as coefficients.] **[4]**

51. M/J 03/P03/Q6

(i) *EITHER* State or imply $f(x) \equiv \dfrac{A}{2x+1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$

State or obtain $A = 1$

State or obtain $C = 8$

Use any relevant method to find B

Obtain value $B = 4$

OR State or imply $f(x) \equiv \dfrac{A}{2x+1}+\dfrac{Dx+E}{(x-2)^2}$

State or obtain $A = 1$

Use any relevant method to find D or E

Obtain value $D = 4$

Obtain value $E = 0$ **[5]**

(ii) *EITHER* Use correct method to obtain the first two terms of the expansion of $(1 + 2x)^{-1}$ or $(x - 2)^{-1}$ or $(x - 2)^{-2}$ or $(1 - ½x)^{-1}$ or $(1 - ½x)^{-2}$

Obtain any correct sum of unsimplified expansions up to the terms in x^2 (deduct A1 for each incorrect expansion)

Obtain the given answer correctly

[Unexpanded binomial coefficients involving -1 or -2, e.g. $\begin{pmatrix}-2\\1\end{pmatrix}$ are not sufficient for the M1.]

[f.t. is on A, B, C, D, E.]

[Apply this scheme to attempts to expand $(9x^2 +4)(1+2x)^{-1}(x-2)^{-2}$, giving M1A2 for a correct product of expansions and A1 for multiplying out and reaching the given answer correctly.]

[Allow attempts to multiply out $(1 + 2x)(x - 2)^2(1 - x + 5x^2)$, giving B1 for reduction to a product of two expressions correct up to their terms in x^2, M1 for attempting to multiply out as far as terms in x^2, A1 for a correct expansion, and A1 for obtaining $9x^2 + 4$ correctly.]

[SR: B or C omitted from the form of partial fractions. In part **(i)** give the first B1, and M1 for the use of a relevant method to obtain A, B, or C, but no further marks. In part **(ii)** only the M1 and A1√ for an unsimplified sum are available.]

[SR: E omitted from the form of partial fractions. In part (i) give the first B1, and M1 for the use of a relevant method to obtain A or D, but no further marks. In part (ii) award M1A2√A1 as in the scheme.]

OR Differentiate and evaluate f(0) and f′(0)
Obtain f(0) = 1 and f′(0) = -1
Differentiate and obtain f″(0) = 10
Form the Maclaurin expansion and obtain the given answer correctly [4]

52. O/N 02/P03/Q6

(i) State or imply $f(x) = \frac{A}{(2-x)} + \frac{Bx+C}{(x^2+1)}$
State or obtain $A = 4$
Use any relevant method to find B or C
Obtain both $B = 4$ and $C = 1$ [4]

(ii) *EITHER*: Use correct method to obtain the first two terms of the expansion of $(1-\frac{1}{2}x)^{-1}$, or $(1+x^2)^{-1}$, or $(2-x)^{-1}$
Obtain unsimplified expansion of the fractions e.g. $\frac{4}{2}(1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3)$; $(4x+1)(1-x^2)$
Carry out multiplication of expansion of $(1+x^2)^{-1}$ by $(4x+1)$
Obtain given answer correctly
[Binomial coefficients involving −1, such as $\begin{pmatrix}-1\\1\end{pmatrix}$, are not sufficient for the first M1.]
[f.t. is on A, B, C.]
[Apply this scheme to attempts to expand $(6+7x)(2-x)^{-1}(1-x^2)^{-1}$, giving M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for reaching the given answer.]

OR: Differentiate and evaluate f(0) and f′(0)
Obtain f(0) = 3 and f′(0) = 5
Differentiate and obtain f″(0) = −1
Differentiate, evaluate f‴(0) and form the Maclaurin expansion up to the term in x^3
Simplify coefficients and obtain given answer correctly [5]
[f.t. is on A, B, C.]

[SR: B or C omitted from the form of partial fractions. In part (i) give the first B1, and M1 for the use of a relevant method to obtain A, B or C, but no further marks. In part (ii) only the first M1 and A1√ + A1√ are available if an attempt is based on this form of partial fractions]

53. M/J 02/P03/Q2

EITHER: Show correct (unsimplified) version of the x or the x^2 or the x^3 term
Obtain correct first two terms $1 + x$
Obtain correct quadratic term $2x^2$
Obtain correct cubic term $\frac{14}{3}x^3$ (allow $\frac{28}{6}$, 4.67, 4.66 for the coefficient)
[The M mark may be implied by correct simplified terms, if no working is shown. It is not earned by unexpanded binomial coefficients involving $-\frac{1}{3}$, e.g. $^{-\frac{1}{3}}C_2$ or $\begin{pmatrix}-\frac{1}{3}\\2\end{pmatrix}$.]

[An attempt to divide 1 by the expansion of $(1-3x)^{\frac{1}{3}}$ earns M1 if the expansion has a correct (unsimplified) x, x^2, or x^3 term and if the partial quotient contains a term in x. The remaining A marks are awarded as above.]

OR: Differentiate and calculate f(0), f′(0), where $f'(x) = k(1-3x)^{-\frac{1}{3}-1}$
Obtain correct first two terms $1 + x$
Obtain correct quadratic term $2x^2$
Obtain correct cubic term $\frac{14}{3}x^3$ (allow $\frac{28}{6}$, 4.67, 4.66 for the coefficient) [4]

UNIT 2

Logarithmic and Exponential Functions

A-Level

Mathematics Paper 3

Topical Workbook

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Unit-2: Logarithmic And Exponential Functions

1. M/J 18/P32/Q1

Showing all necessary working, solve the equation $3|2^x - 1| = 2^x$, giving your answers correct to 3 significant figures. [4]

2. M/J 18/P31/Q1

Showing all necessary working, solve the equation $\ln(x^4 - 4) = 4\ln x - \ln 4$, giving your answer correct to 2 decimal places. [4]

3. M/J 18/P33/Q2

Showing all necessary working, solve the equation $5^{2x} = 5^x + 5$. Give your answer correct to 3 decimal places. [5]

4. O/N 17/P32/Q2

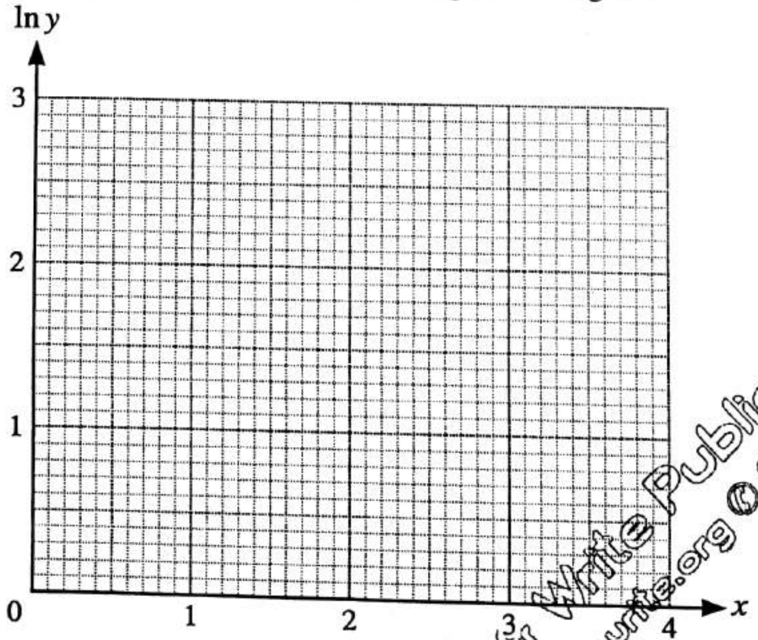
Showing all necessary working, solve the equation $2\log_2 x = 3 + \log_2(x+1)$, giving your answer correct to 3 significant figures. [5]

5. O/N 17/P31/Q2, O/N 17/P33/Q2

Two variable quantities x and y are believed to satisfy an equation of the form $y = C(a^x)$, where C and a are constants. An experiment produced four pairs of values of x and y. The table below gives the corresponding values of x and $\ln y$.

x	0.9	1.6	2.4	3.2
$\ln y$	1.7	1.9	2.3	2.6

By plotting $\ln y$ against x for these four pairs of values and drawing a suitable straight line, estimate the values of C and a. Give your answers correct to 2 significant figures. [5]



6. M/J 17/P32/Q1

Solve the equation $\ln(x^2 + 1) = 1 + 2\ln x$, giving your answer correct to 3 significant figures. [3]

7. M/J 17/P31/Q3(i)

It is given that $x = \ln(1 - y) - \ln y$, where $0 < y < 1$.

(i) Show that $y = \dfrac{e^{-x}}{1 + e^{-x}}$. [2]

8. M/J 17/P33/Q3
Using the substitution $u = e^x$, solve the equation $4e^{-x} = 3e^x + 4$. Give your answer correct to 3 significant figures. **[4]**

9. O/N 16/P32/Q1, O/N 16/P31/Q1
Solve the equation $\dfrac{3^x + 2}{3^x - 2} = 8$, giving your answer correct to 3 decimal places. **[3]**

10. O/N 16/P33/Q1
It is given that $z = \ln(y + 2) - \ln(y + 1)$. Express y in terms of z. **[3]**

11. M/J 16/P32/Q1
Use logarithms to solve the equation $4^{3x-1} = 3(5^x)$, giving your answer correct to 3 decimal places. **[4]**

12. M/J 16/P33/Q2
The variables x and y satisfy the relation $3^y = 4^{2-x}$.

(i) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. **[3]**

(ii) Calculate the exact x-coordinate of the point of intersection of this line with the line with equation $y = 2x$, simplifying your answer. **[2]**

13. O/N 15/P32/Q2, O/N 15/P31/Q2
Using the substitution $u = 3^x$, solve the equation $3^x + 3^{2x} = 3^{3x}$ giving your answer correct to 3 significant figures. **[5]**

14. O/N 15/P33/Q1
Sketch the graph of $y = e^{ax} - 1$ where a is a positive constant. **[2]**

15. M/J 15/P32/Q2
Using the substitution $u = 4^x$, solve the equation $4^x + 4^2 = 4^{x+2}$, giving your answer correct to 3 significant figures. **[4]**

16. M/J 15/P31/Q1
Use logarithms to solve the equation $2^{5x} = 3^{2x+1}$, giving the answer correct to 3 significant figures. **[4]**

17. M/J 15/P33/Q1
Solve the equation $\ln(x + 4) = 2\ln x + \ln 4$, giving your answer correct to 3 significant figures. **[4]**

18. O/N 14/P32/Q1, O/N 14/P31/Q1
Use logarithms to solve the equation $e^x = 3^{x-2}$, giving your answer correct to 3 decimal places. **[3]**

19. M/J 14/P32/Q2
Solve the equation

$$2\ln(5 - e^{-2x}) = 1,$$

giving your answer correct to 3 significant figures. **[4]**

20. M/J 14/P31/Q3
The parametric equations of a curve are

$$x = \ln(2t + 3), \quad y = \frac{3t + 2}{2t + 3}.$$

Find the gradient of the curve at the point where it crosses the y-axis. **[6]**

21. M/J 14/P33/Q1

Solve the equation $\log_{10}(x+9) = 2 + \log_{10} x$. [3]

22. O/N 13/P32/Q2

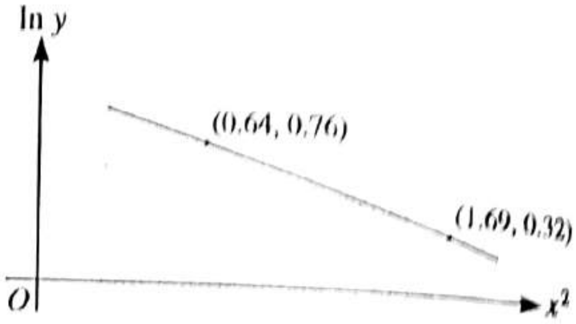
Solve the equation $2|3^x - 1| = 3^x$, giving your answers correct to 3 significant figures. [4]

23. O/N 13/P33/Q1

Given that $2\ln(x+4) - \ln x = \ln(x+a)$, express x in terms of a. [4]

24. M/J 13/P32/Q3

The variables x and y satisfy the equation $y = Ae^{-kx^2}$, where A and k are constants. The graph of $\ln y$ against x^2 is a straight line passing through the points $(0.64, 0.76)$ and $(1.69, 0.32)$, as shown in the diagram. Find the values of A and k correct to 2 decimal places. [5]



25. M/J 13/P31/Q4

(i) Solve the equation $|4x - 1| = |x - 3|$. [3]

(ii) Hence solve the equation $|4^{y+1} - 1| = |4^y - 3|$ correct to 3 significant figures. [3]

26. M/J 13/P33/Q2

It is given that $\ln(y+1) - \ln y = 1 + 3\ln x$. Express y in terms of x, in a form not involving logarithms. [4]

27. O/N 12/P32/Q2, O/N 12/P31/Q2

Solve the equation

$$5^{x-1} = 5^x - 5,$$

giving your answer correct to 3 significant figures. [4]

28. O/N 12/P33/Q1

Solve the equation

$$\ln(x+5) = 1 + \ln x,$$

giving your answer in terms of e. [3]

29. M/J 12/P32/Q1

Solve the equation

$$\ln(3x+4) = 2\ln(x+1),$$

giving your answer correct to 3 significant figures. [4]

30. M/J 12/P33/Q2

Solve the equation $\ln(2x+3) = 2\ln x + \ln 3$, giving your answer correct to 3 significant figures. [4]

31. O/N 11/P32/Q1, O/N 11/P31/Q1

Using the substitution $u = e^x$, or otherwise, solve the equation

$$e^x = 1 + 6e^{-x},$$

giving your answer correct to 3 significant figures. [4]

32. M/J 11/P32/Q2

(i) Show that the equation

$$\log_2(x+5) = 5 - \log_2 x$$

can be written as a quadratic equation in x. [3]

(ii) Hence solve the equation

$$\log_2(x+5) = 5 - \log_2 x.$$ [2]

33. M/J 11/P31/Q5

The curve with equation

$$6e^{2x} + ke^{y} + e^{2y} = c,$$

where k and c are constants, passes through the point P with coordinates $(\ln 3, \ln 2)$.

(i) Show that $58 + 2k = c$. [2]

(ii) Given also that the gradient of the curve at P is -6, find the values of k and c. [5]

34. M/J 11/P33/Q1

Use logarithms to solve the equation $5^{2x-1} = 2(3^x)$, giving your answer correct to 3 significant figures. [4]

35. O/N 10/P32/Q2, O/N 10/P31/Q2

Solve the equation

$$\ln(1+x^2) = 1 + 2\ln x,$$

giving your answer correct to 3 significant figures. [4]

36. M/J 10/P32/Q1

Solve the equation

$$\frac{2^x+1}{2^x-1} = 5,$$

giving your answer correct to 3 significant figures. [4]

37. M/J 10/P31/Q3

The variables x and y satisfy the equation $x^n y = C$, where n and C are constants. When $x = 1.10$, $y = 5.20$, and when $x = 3.20$, $y = 1.05$.

(i) Find the values of n and C. [5]

(ii) Explain why the graph of $\ln y$ against $\ln x$ is a straight line. [1]

38. M/J 10/P33/Q2

The variables x and y satisfy the equation $y^3 = Ae^{2x}$, where A is a constant. The graph of $\ln y$ against x is a straight line.

(i) Find the gradient of this line. [2]

(ii) Given that the line intersects the axis of $\ln y$ at the point where $\ln y = 0.5$, find the value of A correct to 2 decimal places. [2]

39. O/N 09/P32/Q1

Solve the equation

$$\ln(5-x) = \ln 5 - \ln x,$$

giving your answers correct to 3 significant figures. [4]

40. O/N 09/P31/Q2

Solve the equation $3^{x+2} = 3^x + 3^2$, giving your answer correct to 3 significant figures. [4]

41. M/J 09/P3/Q1

Solve the equation $\ln(2 + e^{-x}) = 2$, giving your answer correct to 2 decimal places. [4]

42. O/N 08/P3/Q1

Solve the equation

$$\ln(x+2) = 2 + \ln x,$$

giving your answer correct to 3 decimal places. [3]

43. M/J 08/P3/Q2

Solve, correct to 3 significant figure, the equation [5]

$$e^x + e^{2x} = e^{3x}.$$

44. M/J 07/P3/Q4

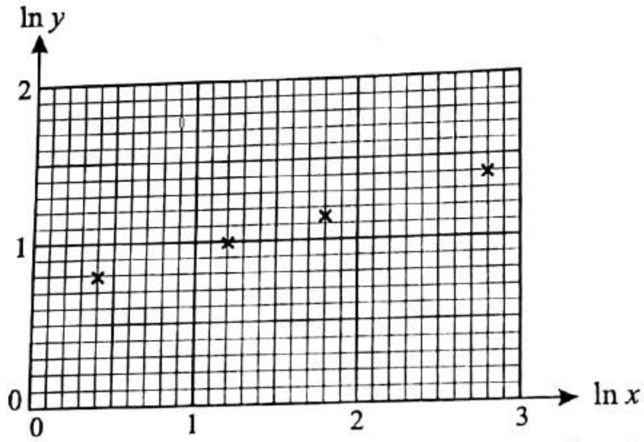
Using the substitution $u = 3^x$, or otherwise, solve, correct to 3 significant figures, the equation [6]

$$3^x = 2 + 3^{-x}.$$

45. M/J 06/P3/Q1

Given that $x = 4(3^{-y})$, express y in terms of x. [3]

46. O/N 05/P3/Q2



Two variable quantities x and y are related by the equation $y = Ax^n$, where A and n are constants. The diagram shows the result of plotting $\ln y$ against $\ln x$ for four pairs of values of x and y. Use the diagram to estimate the values of A and n. [5]

47. O/N 04/P3/Q2

Solve the equation

$$\ln(1+x) = 1 + \ln x,$$

giving your answer correct to 2 significant figures. [4]

48. M/J 04/P3/Q4

(i) Show that if $y = 2^x$, then the equation

$$2^x - 2^{-x} = 1$$

can be written as a quadratic equation in y. [2]

(ii) Hence solve the equation [4]

$$2^x - 2^{-x} = 1.$$

49. O/N 02/P3/Q3

(i) Show that the equation

$$\log_{10}(x+5) = 2 - \log_{10}x$$

may be written as a quadratic equation in x. [3]

(ii) Hence find the value of x satisfying the equation [2]

$$\text{Log}_{10}(x+5) = 2 - \log_{10}x.$$

Answers Section

1. M/J 18/P32/Q1

EITHER: State or imply non-modular equation

$3^2\left(2^x-1\right)^2=\left(2^x\right)^2$, or pair of equations

$3\left(2^x-1\right)=\pm 2^x$

Obtain $2^x=\dfrac{3}{2}$ and $2^x=\dfrac{3}{4}$ or equivalent

OR: Obtain $2^x=\dfrac{3}{2}$ by solving an equation

Obtain $2^x=\dfrac{3}{4}$ by solving an equation

Use correct method for solving an equation of the form $2^x=a$, where $a>0$

Obtain **final** answers $x=0.585$ and $x=-0.415$ only 4

2. M/J 18/P31/Q1

Use law for the logarithm of a product, quotient or power

Obtain a correct equation free of logarithms, e.g. $4(x^4-4)=x^4$

Solve for x

Obtain answer $x=1.52$ only 4

3. M/J 18/P33/Q2

State or imply $u^2=u+5$, or equivalent in 5^x

Solve for u, or 5^x

Obtain root $\dfrac{1}{2}(1+\sqrt{21})$, or decimal in [2.79, 2.80]

Use correct method for finding x from a positive root

Obtain answer $x=0.638$ and no other answer 5

4. O/N 17/P32/Q2

Use law for the logarithm of a power or a quotient on the given equation

Use $\log_2 8=3$ or $2^3=8$

Obtain $x^2-8x-8=0$, or horizontal equivalent

Solve a 3-term quadratic equation

Obtain final answer $x=8.90$ only 5

5. O/N 17/P31/Q2, O/N 17/P33/Q2

Plot the four points and draw straight line

State or imply that $\ln y=\ln C+x\ln a$

Carry out a completely correct method for finding $\ln C$ or $\ln a$

Obtain answer $C=3.7$

Obtain answer $a=1.5$ 5

6. M/J 17/P32/Q1

Use law of the logarithm of a power or a quotient

Remove logarithms and obtain a correct equation in x. e.g. $x^2+1=ex^2$

Obtain answer 0.763 and no other 3

7. M/J 17/P31/Q3(i)

(i) Remove logarithms correctly and obtain $e^x = \dfrac{1-y}{y}$

Obtain the **given answer** $y = \dfrac{e^{-x}}{1+e^{-x}}$ following full working 2

8. M/J 17/P33/Q3

Rearrange as $3u^2 + 4u - 4 = 0$, or $3e^{2x} + 4e^x - 4 = 0$, or equivalent

Solve a 3-term quadratic for e^x or for u

Obtain $e^x = \frac{2}{3}$ or $u = \frac{2}{3}$

Obtain answer $x = -0.405$ and no other 4

9. O/N 16/P32/Q1, O/N 16/P31/Q1

Solve for 3^x and obtain $3^x = \frac{18}{7}$

Use correct method for solving an equation of the form $3^x = a$, where $a > 0$

Obtain answer $x = 0.860$ 3 d.p. only [3]

10. O/N 16/P33/Q1

Use law of the logarithm of a quotient

Remove logarithms and obtain a correct equation, e.g. $e^z = \dfrac{y+2}{y+1}$

Obtain answer $y = \dfrac{2-e^z}{e^z - 1}$, or equivalent [3]

11. M/J 16/P32/Q1

Use law of the logarithm of a product, power or quotient

Obtain a correct linear equation, e.g. $(3x-1)\ln 4 = \ln 3 + x\ln 5$

Solve a linear equation for x

Obtain answer $x = 0.975$ [4]

12. M/J 16/P33/Q2

(i) State or imply $y\ln 3 = (2-x)\ln 4$

State that this is of the form $ay = bx + c$ and thus a straight line, or equivalent

State gradient is $-\dfrac{\ln 4}{\ln 3}$, or exact equivalent [3]

(ii) Substitute $y = 2x$ and solve for x, using a log law correctly at least once

Obtain answer $x = \ln 4 / \ln 6$, or exact equivalent [2]

13. O/N 15/P32/Q2, O/N 15/P31/Q2

State or imply $1 + u = u^2$

Solve for u

Obtain root $\frac{1}{2}(1+\sqrt{5})$, or decimal in [1.61, 1.62]

Use correct method for finding x from a positive root

Obtain $x = 0.438$ and no other answer [5]

14. O/N 15/P33/Q1

Draw curve with increasing gradient existing for negative and positive values of x

Draw correct curve passing through the origin [2]

15. M/J 15/P32/Q2

Use laws of indices correctly and solve for u

Obtain u in any correct form, e.g. $u = \dfrac{16}{16-1}$

Use correct method for solving an equation of the form $4^x = a$, where $a > 0$

Obtain answer $x = 0.0466$ [4]

16. M/J 15/P31/Q1

Use law for the logarithm of a power at least once

Obtain correct linear equation, e.g. $5x\ln 2 = (2x+1)\ln 3$

Solve a linear equation for x

Obtain $x = 0.866$ [4]

17. M/J 15/P33/Q1

Use law for the logarithm of a product, quotient or power

Obtain a correct equation free of logarithms, e.g. $\dfrac{x+4}{x^2} = 4$

Solve a 3-term quadratic obtaining at least one root

Obtain final answer $x = 1.13$ only 4

18. O/N 14/P32/Q1, O/N 14/P31/Q1

Use law of the logarithm of a power

Obtain a correct linear equation in any form, e.g. $x = (x-2)\ln 3$

Obtain answer $x = 22.281$ [3]

19. M/J 14/P32/Q2

Remove logarithms and obtain $5 - e^{-2x} = e^{\frac{1}{2}}$, or equivalent

Obtain a correct value for e^{-2x}, e^{2x}, e^{-x} or e^{x}, e.g. $e^{2x} = 1/(5 - e^{\frac{1}{2}})$

Use correct method to solve an equation of the form $e^{2x} = a$, $e^{-2x} = a$, $e^{x} = a$ or $e^{-x} = a$ where $a > 0$. [The M1 is dependent on the correct removal of logarithms.]

Obtain answer $x = -0.605$ only. 4

20. M/J 14/P31/Q3

Obtain $\dfrac{2}{2t+3}$ for derivative of x

Use quotient of product rule, or equivalent, for derivative of y

Obtain $\dfrac{5}{(2t+3)^2}$ or unsimplified equivalent

Obtain $t = -1$

Use $\dfrac{dy}{dx} = \dfrac{dy}{dt} / \dfrac{dx}{dt}$ in algebraic or numerical form

Obtain gradient $\dfrac{5}{2}$ [6]

21. M/J 14/P33/Q1

Use law of the logarithm of a quotient or product or $2 = \log_{10}100$

Remove logarithms and obtain $x + 9 = 100x$, or equivalent

Obtain answer $x = \frac{1}{11}$ 3

22. O/N 13/P32/Q2

EITHER: State or imply non-modular equation $2^2\left(3^x-1\right)^2=\left(3^x\right)^2$, or pair of equations $2\left(3^x-1\right)=\pm 3^x$

Obtain $3^x=2$ and $3^x=\frac{2}{3}$ (or $3^{x+1}=2$)

OR: Obtain $3^x=2$ by solving an equation or by inspection

Obtain $3^x=\frac{2}{3}$ (or $3^{x+1}=2$) by solving an equation or by inspection

Use correct method for solving an equation of the form $3^x=a$ (or $3^{x+1}=a$), where $a>0$

Obtain final answers 0.631 and −0.369 **[4]**

23. O/N 13/P33/Q1

Apply at least one logarithm property correctly

Obtain $\frac{(x+4)^2}{x}=x+a$ or equivalent **without logarithm** involved

Rearrange to express x in terms of a

Obtain $\frac{16}{a-8}$ or equivalent **[4]**

24. M/J 13/P32/Q3

EITHER: State or imply $\ln y=\ln A-kx^2$

Substitute values of $\ln y$ and x^2, and solve for k or $\ln A$

Obtain $k=0.42$ or $A=2.80$

Solve for $\ln A$ or k

Obtain $A=2.80$ or $k=0.42$

OR1: State or imply $\ln y=\ln A-kx^2$

Using values of $\ln y$ and x^2, equate gradient of line to $-k$ and solve for k

Obtain $k=0.42$

Solve for $\ln A$

Obtain $A=2.80$

OR2: Obtain two correct equations in k and A and substituting y– and x^2 – values in $y=Ae^{-kx^2}$

Solve for k

Obtain $k=0.42$

Solve for A

Obtain $A=2.80$ **[5]**

[SR: If unsound substitutions are made, e.g. using $x=0.;64$ and $y=0.76$, give B1M0A0M1A0 in the *EITHER* and *OR1* schemes, and B0M1A0M1A0 in the *OR2* scheme.]

25. M/J 13/P31/Q4

(i) <u>Either</u> State or imply non-modular equation $(4x-1)^2=(x-3)^2$, or pair of linear equations $4x-1=\pm(x-3)$

Solve a three-term quadratic equation or two linear equations

Obtain $-\frac{2}{3}$ and $\frac{4}{5}$

<u>Or</u> Obtain value $-\frac{2}{3}$ from inspection or solving linear equation

Obtain value $\frac{4}{5}$ similarly **[3]**

(ii) State or imply at least $4^y = \frac{4}{5}$, following a positive answer from part (i)

Apply logarithms and use $\log a^b = b\log a$ property

Obtain -0.161 and no other answer **[3]**

26. M/J 13/P33/Q2

Use law for the logarithm of a product, quotient or power

Use $\ln e = 1$ or $\exp(1) = 3$

Obtain correct equation free of logarithms in any form, e.g. $\frac{y+1}{y} = ex^3$

Rearrange as $y = (ex^3 - 1)^{-1}$, or equivalent **[4]**

27. O/N 12/P32/Q2, O/N 12/P31/Q2

EITHER Use laws of indices correctly and solve for 5^x or for 5^{-x} or for 5^{x-1}

Obtain 5^x or for 5^{-x} or for 5^{x-1} in any correct form, e.g. $5^x = \frac{5}{1-{}^1/_5}$

Use correct method for solving $5^x = a$, or $5^{-x} = a$, or $5^{x-1} = a$, where $a > 0$

Obtain answer $x = 1.14$

OR Use an appropriate iterative formula, e.g. $x_{n+1} = \frac{\ln(5^{x-1}+5)}{\ln 5}$, correctly, at least once

Obtain answer 1.14

Show sufficient iterations to at least 3 d.p. to justify 1.14 to 2 d.p., or show there is a sign change in the interval (1.135, 1.145)

Show there is no other root **[4]**

[For the solution $x = 1.14$ with no relevant working give B1, and a further B1 if 1.14 is shown to be the only solution.]

28. O/N 12/P33/Q1

State or imply $\ln e = 1$

Apply at least one logarithm law for product or quotient correctly (or exponential equivalent)

Obtain $x + 5 = ex$ or equivalent and hence $\frac{5}{e-1}$ **[3]**

29. M/J 12/P32/Q1

EITHER: Use law of the logarithm of a power or quotient and remove logarithms

Obtain a 3-term quadratic equation $x^2 - x - 3 = 0$, or equivalent

Solve 3-term quadratic obtaining 1 or 2 roots

Obtain answer 2.30 only

OR1: Use an appropriate iterative formula, e.g. $x_{n+1} = \exp\left(\frac{1}{2}\ln(3x_n + 4)\right) - 1$ correctly at least once

Obtain answer 2.30

Show sufficient iterations to at least 3 d.p. to justify 2.30 to 2 d.p., or show there is a sign change in the interval (2.295, 2.305)

Show there is no other root

OR2: Use calculated values to obtain at least one interval containing the root

Obtain answer 2.30

Show sufficient calculations to justify 2.30 to 3 s.f., e.g. show it lies in (2.295, 2.305)

Show there is no other root **[4]**

30. M/J 12/P33/Q2

Use law of the logarithm of a power and a product or quotient and remove logarithms

Obtain a correct equation in any form, e.g. $\dfrac{2x+3}{x^2}=3$

Solve 3-term quadratic obtaining at least one root

Obtain final answer 1.39 only **[4]**

31. O/N 11/P32/Q1, O/N 11/P31/Q1

Rearrange as $e^{2x}-e^x-6=0$, or $u^2-u-6=0$, or equivalent

Solve a 3-term quadratic for e^x or for u

Obtain simplified solution $e^x=3$ or $u=3$

Obtain final answer $x=1.10$ and no other **[4]**

32. M/J 11/P32/Q2

(i) Use law for the logarithm of a product or quotient

Use $\log_2 32=5$ or $2^5=32$

Obtain $x^2+5x-32=0$, or horizontal equivalent **[3]**

(ii) Solve a 3-term quadratic equation

Obtain answer $x=3.68$ only, or exact equivalent, e.g. $\dfrac{\sqrt{153}-5}{2}$ **[2]**

33. M/J 11/P31/Q5

(i) Use at least one of $e^{2x}=9$, $e^y=2$ and $e^{2y}=4$

Obtain given result $58+2k=c$ **[2]**

(ii) Differentiate left-hand side term by term, reaching $ae^{2x}+be^y\dfrac{dy}{dx}+ce^{2y}\dfrac{dy}{dx}$

Obtain $12e^{2x}+ke^y\dfrac{dy}{dx}+2e^{2y}\dfrac{dy}{dx}$

Substitute (ln 3, ln 2) in an attempt involving implicit differentiation at least once, where RHS = 0

Obtain $108-12k-48=0$ or equivalent

Obtain $k=5$ and $c=68$ **[5]**

34. M/J 11/P33/Q1

Use law for the logarithm of a product, power or quotient

Obtain a correct linear equation, e.g. $(2x-1)\ln 5=\ln 2+x\ln 3$

Solve a linear equation for x

Obtain answer $x=1.09$ **[4]**

[SR: Reduce equation to the form $a^x=b$ M1*, obtain $\left(\dfrac{25}{3}\right)^x=10$ A1, use correct method to calculate value of x M1(dep*), obtain answer 1.09 A1.]

35. O/N 10/P32/Q2, O/N 10/P31/Q2

Use law for the logarithm of a power, a quotient, or a product correctly at least once

Use $\ln e=1$ or $e=\exp(1)$

Obtain a correct equation free of logarithms, e.g. $1+x^2=ex^2$

Solve and obtain answer $x=0.763$ only **[4]**

[For the solution $x=0.763$ with no relevant working give B1, and a further B1 if 0.763 is shown to be the only root.]

[Treat the use of logarithms to base 10 with answer 0.333 only, as a misread.]

[SR: Allow iteration, giving B1 for an appropriate formula, e.g. $x_{n+1}=\exp((\ln(1+x_n^2)-1)/2)$, M1 for using it correctly once, A1 for 0.763, and A1 for showing the equation has no other root but 0.763.]

36. M/J 10/P32/Q1

EITHER: Attempt to solve for 2^x

Obtain $2^x = 6/4$, or equivalent
Use correct method for solving an equation of the form $2^x = a$, where $a > 0$
Obtain answer $x = 0.585$

OR: State an appropriate iterative formula, e.g. $x_{n+1} = \ln((2^{x_n} + 6) / 5) / \ln 2$
Use the iterative formula correctly at least once
Obtain answer $x = 0.585$
Show that the equation has no other root but 0.585 **[4]**

[For the solution 0.585 with no relevant working, award B1 and a further B1 if 0.585 is shown to be the only root.]

37. M/J 10/P31/Q3

(i) *EITHER*: State or imply $n \ln x + \ln y = \ln C$
Substitute x- and y-values and solve for n
Obtain $n = 1.50$
Solve for C
Obtain $C = 6.00$

OR: Obtain two correct equations by substituting x- and y-values in $x^n y = C$
Solve for n
Obtain $n = 1.50$
Solve for C
Obtain $C = 6.00$ **[5]**

(ii) State that the graph of $\ln y$ against $\ln x$ has equation $n\ln x + \ln y = \ln C$ which is *linear* in $\ln y$ and $\ln x$, or has equation of the form $nX + Y = \ln C$, where $X = \ln x$ and $Y = \ln y$, and is thus a straight line **[1]**

38. M/J 10/P33/Q2

(i) State or imply $3 \ln y = \ln A + 2x$ at any stage
State gradient is $\frac{2}{3}$, or equivalent **[2]**

(ii) Substitute $x = 0$, $\ln y = 0.5$ and solve for A
Obtain $A = 4.48$ **[2]**

39. O/N 09/P32/Q1

Use law of the logarithm of a product or quotient and remove logarithms
Obtain quadratic equation $x^2 - 5x + 5 = 0$, or equivalent
Solve 3-term quadratic obtaining 1 or 2 roots
Obtain answers 1.38 and 3.62 **[4]**

40. O/N 09/P31/Q2

EITHER: Use laws of indices correctly and solve a linear equation for 3^x, or for 3^{-x}

Obtain 3^x, or 3^{-x} in any correct form, e.g. $3^x = \frac{3^2}{(3^2 - 1)}$

Use correct method for solving $3^{\pm x} = a$ for x, where $a > 0$
Obtain answer $x = 0.107$

OR: State an appropriate iterative formula, e.g. $x_{n+1} = \frac{\ln(3^{x_n} + 9)}{\ln 3} - 2$

Use the formula correctly at least once
Obtain answer $x = 0.107$
Show that the equation has no other root but 0.107 **[4]**

[For the solution 0.107 with no relevant working, award B1 and a further B1 if 0.107 is shown to be the only root.]

41. M/J 09/P3/Q1

State or imply $2 + e^{-x} = e^2$
Carry out method for finding $\pm x$ from $e^{\pm x} = k$, where $k > 0$, following sound ln or exp work
Obtain $x = -\ln(e^2 - 2)$, or equivalent expression for x
Obtain answer $x = -1.68$ 4
[The answer must be given to 2 decimal places]
[SR: the M1 is available for attempts starting with $2 + e^{-x} = 10^2$]

42. O/N 08/P3/Q1

Use laws of logarithms and remove logarithms correctly
Obtain $x + 2 = e^2 x$, or equivalent
Obtain answer $x = 0.313$ [3]
[SR: If the logarithmic work is to base 10 then only the M mark is available.]

43. M/J 08/P3/Q2

EITHER State or imply $e^x + 1 = e^{2x}$, or $1 + e^{-x} = e^x$, or equivalent
Solve this equation as a quadratic in $u = e^x$, or in e^x, obtaining one or two roots
Obtain root $\frac{1}{2}(1 + \sqrt{5})$, or decimal in [1.61, 1.62]
Use correct method for finding x from a positive root
Obtain $x = 0.481$ and no other answer
[For the solution 0.481 with no working, award B3 (for 0.48 give B2). However a suitable statement can earn the first B1 in addition, giving a maximum of 4/5 (or 3/5) in such cases.]

OR State an appropriate iterative formula, e.g. $x_{n+1} = \frac{1}{2}\ln\left(1 + e^{x_n}\right)$ or $x_{n+1} = \frac{1}{3}\ln\left(e^{x_n} + e^{2x_n}\right)$
Use the iterative formula correctly at least once
Obtain final answer 0.481
Show sufficient iterations to justify its accuracy to 3 d.p., or show there is a sign change in the value of a relevant function in the interval (0.4805, 0.4815)
Show that the equation has no other root [5]

44. M/J 07/P3/Q4

Sate or imply at any stage that $3^{-x} = \dfrac{1}{3^x}$, or that $3^{-x} = \dfrac{1}{u}$ where $u = 3^x$
Convert given equation into the 3-term quadratic in u (or 3^x): $u^2 - 2u - 1 = 0$
Solve a 3-term quadratic, obtaining one or two roots
Obtain root $\dfrac{2 + \sqrt{8}}{2}$, or a simpler equivalent, or decimal value in [2.40, 2.42]
Use a correct method for finding the value of x from a positive root
Obtain $x = 0.802$ only 6

45. M/J 06/P3/Q1

Use law for the logarithm of a product or quotient, or the logarithm of a power
Obtain $\ln x = \ln 4 - y \ln 3$, or equivalent
Obtain answer $y = \dfrac{\ln 4 - \ln x}{\ln 3}$, or equivalent 3

46. O/N 05/P3/Q2

State or imply that $\ln y = \ln A + n \ln x$
Equate estimate of $\ln y$ -intercept to $\ln A$
Obtain value A between 1.97 and 2.03
Calculate the gradient of the line of data points
Obtain value $n = 0.25$, or equivalent **[5]**

47. O/N 04/P3/Q2

Use law for subtraction or addition of logarithms, or the equivalent in exponentials
Use ln e = 1 or e = exp(1)
Obtain a correct equation free of logarithms e.g. $\frac{1+x}{x} = e$ or $1 + x = ex$
Obtain answer $x = 0.58$ (allow 0.582 or answer rounding to it) **4**

48. M/J 04/P3/Q4

(i) State or imply $2^{-x} = \frac{1}{y}$

Obtain 3-term quadratic e.g. $y^2 - y - 1 = 0$ **2**

(ii) Solve a 3-term quadratic, obtaining 1 or 2 roots

Obtain answer $y = (1+\sqrt{5})/2$, or equivalent

Carry out correct method for solving an equation of the form $2^x = a$, where $a > 0$, reaching a ratio of logarithms

Obtain answer $x = 0.694$ only **4**

49. O/N 02/P3/Q3

(i) Use law for addition (or subtraction) of logarithms or indices

Use $\log_{10} 100 = 2$ or $10^2 = 100$

Obtain $x^2 + 5x = 100$, or equivalent, correctly **3**

(ii) Solve a three –term quadratic equation

State answer 7.81 (allow 7.80 or 7.8) or any exact form of the answer i.e. $\frac{\sqrt{425}-5}{2}$ or better **2**

UNIT 3

Trigonometry

A-Level

Mathematics Paper 3

Topical Workbook

READ & WRITE
PUBLICATIONS

Sale Point: Shop # 25-28 Lower Ground Floor,
Haadia Haleema Centre, Ghazni Street,
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Tel: +92 42-35714038
Cell: +92 336-5314141
Web: www.readnwrite.org
E-mail: readandwriteoffice@gmail.com

Unit 3: Trignometry

M/J 18/P32/Q2

Showing all necessary working, solve the equation $\cot\theta + \cot(\theta + 45°) = 2$, for $0° < \theta < 180°$. [5]

M/J 18/P31/Q2

(i) Given that $\sin(x - 60°) = 3\cos(x - 45°)$, find the exact value of $\tan x$. [4]

(ii) Hence solve the equation $\sin(x - 60°) = 3\cos(x - 45°)$, for $0° < x < 360°$. [2]

M/J 18/P33/Q5

(i) By first expanding $(\cos^2 x + \sin^2 x)^3$, or otherwise, show that

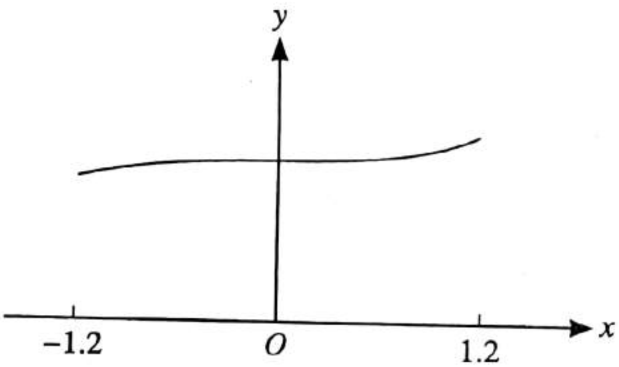
$$\cos^6 x + \sin^6 x = 1 - \tfrac{3}{4}\sin^2 2x .$$ [4]

(ii) Hence solve the equation

$$\cos^6 x + \sin^6 x = \tfrac{2}{3},$$

for $0° < x < 180°$. [4]

4. O/N 17/P32/Q1



The diagram shows a sketch of the curve $y = \dfrac{3}{\sqrt{(9 - x^3)}}$ for values of x from -1.2 to 1.2.

(i) Use the trapezium rule, with two intervals, to estimate the value of

$$\int_{-1.2}^{1.2} \frac{3}{\sqrt{(9 - x^3)}}\,dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) Explain, with reference to the diagram, why the trapezium rule may be expected to give a good approximation to the true value of the integral in this case. [1]

O/N 17/P32/Q3

By expressing the equation $\tan(\theta + 60°) + \tan(\theta - 60°) = \cot\theta$ in terms of $\tan\theta$ only, solve the equation for $0° < \theta < 90°$. [5]

O/N 17/P31/Q4,O/N 17/P33/Q4

(i) Prove the identity $\tan(45° + x) + \tan(45° - x) \equiv 2\sec 2x$. [4]

(ii) Hence sketch the graph of $y = \tan(45° + x) + \tan(45° - x)$ for $0° \leqslant x \leqslant 90°$. [3]

M/J 17/P32/Q3

(i) Express the equation $\cot\theta - 2\tan\theta = \sin 2\theta$ in the form $a\cos^4\theta + b\cos^2\theta + c = 0$, where a, b and c are constants to be determined. [3]

(ii) Hence solve the equation $\cot\theta - 2\tan\theta = \sin 2\theta$ for $90° < \theta < 180°$. **[2]**

8. M/J 17/P32/Q7

(i) Prove that if $y = \dfrac{1}{\cos\theta}$ then $\dfrac{dy}{d\theta} = \sec\theta\tan\theta$. **[2]**

(ii) Prove the identity $\dfrac{1+\sin\theta}{1-\sin\theta} \equiv 2\sec^2\theta + 2\sec\theta\tan\theta - 1$. **[3]**

(iii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{1+\sin\theta}{1-\sin\theta}\,d\theta$. **[4]**

9. M/J 17/P31/Q8

(i) By first expanding $2\sin(x-30°)$, express $2\sin(x-30°) - \cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. **[5]**

(ii) Hence solve the equation

$$2\sin(x-30°) - \cos x = 1,$$

for $0° < x < 180°$. **[3]**

10. M/J 17/P33/Q1

Prove the identity $\dfrac{\cot x - \tan x}{\cot x + \tan x} \equiv \cos 2x$. **[3]**

11. O/N 16/P32/Q3, O/N 16/P31/Q3

Express the equation $\sec\theta = 3\cos\theta + \tan\theta$ as a quadratic equation in $\sin\theta$. Hence solve this equation for $-90° < \theta < 90°$. **[5]**

12. O/N 16/P33/Q2

The equation of a curve is $y = \dfrac{\sin x}{1+\cos x}$, for $-\pi < x < \pi$. Show that the gradient of the curve is positive for all x in the given interval. **[4]**

13. O/N 16/P33/Q3

Express the equation $\cot 2\theta = 1 + \tan\theta$ as a quadratic equation in $\tan\theta$. Hence solve this equation for $0° < \theta < 180°$. **[6]**

14. M/J 16/P32/Q5

(i) Prove the identity $\cos 4\theta - 4\cos 2\theta \equiv 8\sin^4\theta - 3$. **[4]**

(ii) Hence solve the equation

$$\cos 4\theta = 4\cos 2\theta + 3,$$

for $0° \leqslant \theta \leqslant 360°$. **[4]**

15. M/J 16/P31/Q3

By expressing the equation $\operatorname{cosec}\theta = 3\sin\theta + \cot\theta$ in terms of $\cos\theta$ only, solve the equation for $0° < \theta < 180°$. **[5]**

16. M/J 16/P33/Q3

(i) Express $(\sqrt{5})\cos x + 2\sin x$ in the form $R\cos(x-\alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. **[3]**

(ii) Hence solve the equation

$$(\sqrt{5})\cos\tfrac{1}{2}x + 2\sin\tfrac{1}{2}x = 1.2,$$

for $0° < x < 360°$. **[3]**

17. O/N 15/P32/Q3, O/N 15/P31/Q3

The angles θ and ϕ lie between $0°$ and $180°$, and are such that

$$\tan(\theta - \phi) = 3 \quad \text{and} \quad \tan\theta + \tan\phi = 1.$$

Find the possible values of θ and ϕ. **[6]**

18. O/N 15/P33/Q6

The angles A and B are such that

$$\sin(A + 45°) = (2\sqrt{2})\cos A \quad \text{and} \quad 4\sec^2 B + 5 = 12\tan B.$$

Without using a calculator, find the exact value of $\tan(A - B)$. [8]

19. M/J 15/P32/Q4

(i) Express $3\sin\theta + 2\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, stating the exact value of R and giving the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$3\sin\theta + 2\cos\theta = 1,$$

for $0° < \theta < 180°$. [3]

20. M/J 15/P33/Q3

Solve the equation $\cot 2x + \cot x = 3$ for $0° < x < 180°$. [6]

21. O/N 14/P32/Q8,O/N 14/P31/Q8

(i) By first expanding $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$ [4]

(ii) Show that, after making the substitution $x = \dfrac{2\sin\theta}{\sqrt{3}}$, the equation $x^3 - x + \frac{1}{6}\sqrt{3} = 0$ can be written in the form $\sin 3\theta = \frac{3}{4}$. [1]

(iii) Hence solve the equation

$$x^3 - x + \tfrac{1}{6}\sqrt{3} = 0,$$

giving your answers correct to 3 significant figures. [4]

22. O/N 14/P33/Q4

(i) Show that $\cos(\theta - 60°) + \cos(\theta + 60°) \equiv \cos\theta$. [3]

(ii) Given that $\dfrac{\cos(2x - 60°) + \cos(2x + 60°)}{\cos(x - 60°) + \cos(x + 60°)} = 3$, find the exact value of $\cos x$. [4]

23. M/J 14/P32/Q3

Solve the equation

$$\cos(x + 30°) = 2\cos x,$$

giving all solutions in the interval $-180° < x < 180°$. [5]

24. M/J 14/P31/Q1

(i) Simplify $\sin 2\alpha \sec\alpha$. [2]

(ii) Given that $3\cos 2\beta + 7\cos\beta = 0$, find the exact value of $\cos\beta$. [3]

25. M/J 14/P33/Q3

(i) Show that the equation

$$\tan(x - 60°) + \cot x = \sqrt{3}$$

can be written in the form

$$2\tan^2 x + (\sqrt{3})\tan x - 1 = 0.$$ [3]

(ii) Hence solve the equation

$$\tan(x - 60°) + \cot x = \sqrt{3},$$

for $0° < x < 180°$. [3]

26. O/N 13/P33/Q7

(i) Given that $\sec\theta + 2\operatorname{cosec}\theta = 3\operatorname{cosec} 2\theta$, show that $2\sin\theta + 4\cos\theta = 3$. [3]

(ii) Express $2\sin\theta + 4\cos\theta$ in the form $R\sin(\theta + \alpha)$ where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. [3]

(iii) Hence solve the equation $\sec\theta + 2\operatorname{cosec}\theta = 3\operatorname{cosec} 2\theta$ for $0° < \theta < 360°$. [4]

27. M/J 13/P32/Q7

(i) By first expanding $\cos(x + 45°)$, express $\cos(x + 45°) - (\sqrt{2})\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of R correct to 4 significant figures and the value of α correct to 2 decimal places. [5]

(ii) Hence solve the equation

$$\cos(x + 45°) - (\sqrt{2})\sin x = 2,$$

for $0° < x < 360°$. [4]

28. M/J 13/P33/Q3

Solve the equation $\tan 2x = 5\cot x$, for $0° < x < 180°$. [5]

29. O/N 12/P32/Q3, O/N 12/P31/Q3

Solve the equation

$$\sin(\theta + 45°) = 2\cos(\theta - 30°),$$

giving all solutions in the interval $0° < \theta < 180°$. [5]

30. O/N 12/P33/Q2

(i) Express $24\sin\theta - 7\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence find the smallest positive value of θ satisfying the equation

$$24\sin\theta - 7\cos\theta = 17.$$ [2]

31. M/J 12/P32/Q4

Solve the equation

$$\operatorname{cosec} 2\theta = \sec\theta + \cot\theta,$$

giving all solutions in the interval $0° < \theta < 360°$. [6]

32. M/J 12/P32/Q6

The equation of a curve is $y = 3\sin x + 4\cos^3 x$.

(i) Find the x-coordinates of the stationary points of the curve in the interval $0 < x < \pi$. [6]

(ii) Determine the nature of the stationary point in this interval for which x is least. [2]

33. M/J 12/P33/Q6

It is given that $\tan 3x = k\tan x$, where k is a constant and $\tan x \neq 0$.

(i) By first expanding $\tan(2x + x)$, show that

$$(3k - 1)\tan^2 x = k - 3.$$ [4]

(ii) Hence solve the equation $\tan 3x = k\tan x$ when $k = 4$, giving all solutions in the interval $0° < x < 180°$. [3]

(iii) Show that the equation $\tan 3x = k\tan x$ has no root in the interval $0° < x < 180°$ when $k = 2$. [1]

34. O/N 11/P32/Q6

(i) Express $\cos x + 3\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $\cos 2\theta + 3\sin 2\theta = 2$, for $0° < \theta < 90°$. [5]

35. O/N 11/P31/Q2

The parametric equations of a curve are

$$x = 3(1 + \sin^2 t), \quad y = 2\cos^3 t.$$

Find $\frac{dy}{dx}$ in terms of t, simplifying your answer as far as possible. [5]

36. O/N 11/P31/Q6

(i) Express $\cos x + 3\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $\cos 2\theta + 3\sin 2\theta = 2$, for $0° < \theta < 90°$. [5]

37. O/N 11/P33/Q3

(i) Express $8\cos\theta + 15\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $8\cos\theta + 15\sin\theta = 12$, giving all solutions in the interval $0° < \theta < 360°$. [4]

38. M/J 11/P32/Q3

Solve the equation

$$\cos\theta + 4\cos 2\theta = 3,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 180°$. [5]

39. M/J 11/P33/Q4

(i) Show that the equation

$$\tan(60° + \theta) + \tan(60° - \theta) = k$$

can be written in the form

$$(2\sqrt{3})(1 + \tan^2\theta) = k(1 - 3\tan^2\theta).$$ [4]

(ii) Hence solve the equation

$$\tan(60° + \theta) + \tan(60° - \theta) = 3\sqrt{3},$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 180°$. [3]

40. O/N 10/P32/Q3, O/N 10/P31/Q3

Solve the equation

$$\cos(\theta + 60°) = 2\sin\theta,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. [5]

41. O/N 10/P33/Q8

(i) Express $(\sqrt{6})\cos\theta + (\sqrt{10})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence, in each of the following cases, find the smallest positive angle θ which satisfies the equation

(a) $(\sqrt{6})\cos\theta + (\sqrt{10})\sin\theta = -4$, [2]

(b) $(\sqrt{6})\cos\frac{1}{2}\theta + (\sqrt{10})\sin\frac{1}{2}\theta = 3$. [4]

42. M/J 10/P32/Q3

It is given that $\cos a = \frac{3}{5}$, where $0° < a < 90°$. Showing your working and without using a calculator to evaluate a,

(i) find the exact value of $\sin(a - 30°)$, [3]

(ii) find the exact value of $\tan 2a$, and hence find the exact value of $\tan 3a$. [4]

43. M/J 10/P31/Q2

Solve the equation

$$\sin\theta = 2\cos 2\theta + 1,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. [6]

44. M/J 10/P33/Q3

Solve the equation

$$\tan(45° - x) = 2\tan x,$$

giving all solutions in the interval $0° < x < 180°$. [5]

45. O/N 09/P32/Q4

The angles α and β lie in the interval $0° < x < 180°$, and are such that

$$\tan\alpha = 2\tan\beta \quad \text{and} \quad \tan(\alpha + \beta) = 3.$$

Find the possible values of α and β. [6]

46. M/J 09/P3/Q3

(i) Prove the identity $\operatorname{cosec} 2\theta + \cot 2\theta \equiv \cot\theta$. [3]

(ii) Hence solve the equation $\operatorname{cosec} 2\theta + \cot 2\theta = 2$, for $0° \leqslant \theta \leqslant 360°$. [2]

47. M/J 09/P3/Q6

The parametric equations of a curve are

$$x = a\cos^3 t, \quad y = a\sin^3 t,$$

where a is a positive constant and $0 < t < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of t. [3]

(ii) Show that the equation of the tangent to the curve at the point with parameter t is

$$x\sin t + y\cos t = a\sin t\cos t.$$ [3]

(iii) Hence show that, if this tangent meets the x-axis at X and the y-axis at Y, then the length of XY is always equal to a. [2]

48. O/N 08/P3/Q6

(i) Express $5\sin x + 12\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$5\sin 2\theta + 12\cos 2\theta = 11,$$

giving all solutions in the interval $0° < \theta < 180°$. [5]

49. M/J 08/P3/Q4

(i) Show that the equation $\tan(30° + \theta) = 2\tan(60° - \theta)$ can be written in the form

$$\tan^2\theta + (6\sqrt{3})\tan\theta - 5 = 0.$$ [4]

(ii) Hence, or otherwise, solve the equation

$$\tan(30° + \theta) = 2\tan(60° - \theta),$$

for $0° \leqslant \theta \leqslant 180°$. [3]

50. O/N 07/P3/Q5

(i) Show that the equation

$$\tan(45° + x) - \tan x = 2$$

can be written in the form

$$\tan^2 x + 2\tan x - 1 = 0.$$ [3]

(ii) Hence solve the equation

$$\tan(45° + x) - \tan x = 2,$$

giving all solutions in the interval $0° \leqslant x \leqslant 180°$. [4]

51. O/N 06/P3/Q2

Solve the equation

$$\tan x \tan 2x = 1,$$

giving all solutions in the interval $0° < x < 180°$. [4]

52. M/J 06/P3/Q4

(i) Express $7\cos\theta + 24\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$7\cos\theta + 24\sin\theta = 15,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. [4]

53. O/N 05/P3/Q5

By expressing $8\sin\theta - 6\cos\theta$ in the form $R\sin(\theta - \alpha)$, solve the equation

$$8\sin\theta - 6\cos\theta = 7,$$

for $0° \leqslant \theta \leqslant 360°$. [7]

54. M/J 05/P3/Q6

(i) Prove the identity

$$\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3.$$ [4]

(ii) Hence solve the equation

$$\cos 4\theta + 4\cos 2\theta = 2,$$

for $0° \leqslant \theta \leqslant 360°$. [4]

55. O/N 04/P3/Q4

(i) Show that the equation

$$\tan(45^\circ + x) = 2\tan(45^\circ - x)$$

can be written in the form

$$\tan^2 x - 6\tan x + 1 = 0.$$

[4]

(ii) Hence solve the equation $\tan(45^\circ + x) = 2\tan(45^\circ - x)$, for $0^\circ < x < 90^\circ$. [3]

56. M/J 04/P3/Q1

Sketch the graph of $y = \sec x$, for $0 \leqslant x \leqslant 2\pi$. [3]

57. O/N 03/P3/Q3

Solve the equation

$$\cos\theta + 3\cos 2\theta = 2,$$

giving all solutions in the interval $0^\circ \leqslant \theta \leqslant 180^\circ$. [5]

58. M/J 03/P3/Q1

(i) Show that the equation

$$\sin(x - 60^\circ) - \cos(30^\circ - x) = 1$$

can be written in the form $\cos x = k$, where k is a constant. [2]

(ii) Hence solve the equation, for $0^\circ < x < 180^\circ$. [2]

59. O/N 02/P3/Q5

(i) Express $4\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, stating the value of α correct to 2 decimal places. [3]

Hence

(ii) solve the equation

$$4\sin\theta - 3\cos\theta = 2,$$

giving all values of θ such that $0^\circ < \theta < 360^\circ$, [4]

(iii) write down the greatest value of $\dfrac{1}{4\sin\theta - 3\cos\theta + 6}$ [1]

60. M/J 02/P3/Q1

Prove the identity

$$\cot\theta - \tan\theta \equiv 2\cot 2\theta$$

[3]

Answers Section

1. M/J 18/P32/Q2

Use correct tan $(A \pm B)$ formula and obtain an equation in tan θ
Obtain a correct equation in any form
Reduce to $3\tan^2\theta = 1$, or equivalent
Obtain answer $x = 30°$
Obtain answer $x = 150°$
OR: use correct $\sin(A \pm B)$ and $\cos(A \pm B)$ to form equation in $\sin\theta$ and $\cos\theta$
Reduce to $\tan^2\theta = \frac{1}{3}, \sin^2\theta = \frac{1}{4}, \cos^2\theta = \frac{3}{4}$ or $\cot^2\theta = 3$ etc. **5**

2. M/J 18/P31/Q2

(i) Use trig formulae and obtain an equation in sin x and cos x
Obtain a correct equation in any form
Substitute exact trig ratios and obtain an expression for tan x
Obtain answer $\tan x = \frac{-(6+\sqrt{6})}{(6-\sqrt{2})}$ or equivalent **4**

(ii) State answer, e.g. 118.5° **2**
State second answer, e.g. 298.5°

3. M/J 18/P33/Q5

(i) Attempt cubic expansion and equate to 1
Obtain a correct equation
Use Pythagoras and double angle formula in the expansion
Obtain the given result correctly **4**

(ii) Use the identity and carry out a method for finding a root
Obtain answer 20.9°
Obtain a second answer, e.g. 69.1°
Obtain the remaining answers, e.g. 110.9° and 159.1°, and no others in the given interval **4**

4. O/N 17/P32/Q1

(i) State or imply ordinates 0.915929…, 1, 1.112485…
Use correct formula, or equivalent, with $h = 1.2$ and three ordinates
Obtain answer 2.42 only **3**

(ii) Justify the given statement **1**

5. O/N 17/P32/Q3

Use correct $\tan(A \pm B)$ formula and express LHS in terms of tan θ
Using $\tan 60° = \sqrt{3}$ and $\cot\theta = 1/\tan\theta$, obtain a correct equation in $\tan\theta$ in any form
Reduce the equation to one in $\tan^2\theta$ only
Obtain $11\tan^2\theta = 1$, or equivalent
Obtain answer 16.8° **5**

6. O/N 17/P31/Q4, O/N 17/P33/Q4

(i) Use correct $\tan(A \pm B)$ formula and express the LHS in terms of tan x
Using $\tan 45° = 1$ express LHS as a single fraction

Use Pythagoras or correct double angle formula
Obtain given answer 4

(ii) Show correct sketch for one branch
Both branches correct and nothing else seen in the interval
Show asymptote at $x = 45°$ 3

7. M/J 17/P32/Q3

(i) Use correct formulae to express the equation in terms of $\cos\theta$ and $\sin\theta$
Use Pythagoras and express the equation in terms of $\cos\theta$ only
Obtain correct 3-term equation, e.g. $2\cos^4\theta + \cos^2\theta - 2 = 0$ 3

(ii) Solve a 3-term quadratic in $\cos^2\theta$ for $\cos\theta$
Obtain answer $\theta = 152.1°$ only 2

8. M/J 17/P32/Q7

(i) Use quotient or chain rule
Obtain given answer correctly 2

(ii) *EITHER*:
Multiply numerator and denominator of LHS by $1+\sin\theta$
Use Pythagoras and express LHS in terms of $\sec\theta$ and $\tan\theta$
Complete the proof
OR1:
Express RHS in terms of $\cos\theta$ and $\sin\theta$
Use Pythagoras and express RHS in terms of $\sin\theta$
Complete the proof
OR2:
Express LHS in terms of $\sec\theta$ and $\tan\theta$
Multiply numerator and denominator by $\sec\theta + \tan\theta$ and use Pythagoras
Complete the proof 3

(iii) Use the identity and obtain integral $2\tan\theta + 2\sec\theta - \theta$
Use correct limits correctly in an integral containing terms $a\tan\theta$ and $b\sec\theta$
Obtain answer $2\sqrt{2} - \frac{1}{4}\pi$ 4

9. M/J 17/P31/Q8

(i) Use $\sin(A-B)$ formula and obtain an expression in terms of $\sin x$ and $\cos x$
Collect terms and reach $\sqrt{3}\sin x - 2\cos x$, or equivalent
Obtain $R = \sqrt{7}$
Use trig formula to find α
Obtain $\alpha = 49.11°$ with no errors seen 5

(ii) Evaluate $\sin^{-1}(1/\sqrt{7})$ to at least 1 d.p. (22.21° to 2 d.p.)
Use a correct method to find a value of x in the interval $0° < x < 180°$
Obtain answer 71.3°
[ignore answers outside given range.] 3

10. M/J 17/P33/Q1

Express the LHS in terms of either cos x and sin x or in terms of tan x
Use Pythagoras
Obtain the given answer 3

11. O/N 16/P32/Q3, O/N 16/P31/Q3

EITHER: Correctly restate the equation in terms of $\sin\theta$ and $\cos\theta$
Correct method to obtain a horizontal equation in $\sin\theta$
Reduce the equation to a correct quadratic in any form, e.g. $3\sin^2\theta - \sin\theta - 2 = 0$
Solve a three-term quadratic for $\sin\theta$

Obtain final answer $\theta = -41.8°$ only
[Ignore answers outside the given interval.]

OR 1: Square both sides of the equation and use $1+\tan^2\theta = \sec^2\theta$
Correct method to obtain a horizontal equation in $\sin\theta$
Reduce the equation to a correct quadratic in any form, e.g. $9\sin^2\theta - 6\sin\theta - 8 = 0$
Solve a three-term quadratic for $\sin\theta$
Obtain final answer $\theta = -41.8°$ only

OR 2: Multiply through by $(\sec\theta + \tan\theta)$
Use $\sec^2\theta - \tan^2\theta = 1$
Obtain $1 = 3 + 3\sin\theta$
Solve for $\sin\theta$
Obtain final answer $\theta = -41.8°$only [5]

12. O/N 16/P33/Q2

Use correct quotient or product rule
Obtain correct derivative in any form
Use Pythagoras to simplify the derivative to $\dfrac{1}{1+\cos x}$, or equivalent
Justify the given statement, $-1 < \cos x < 1$ statement, or equivalent [4]

13. O/N 16/P33/Q3

Use the tan $2A$ formula to obtain an equation in $\tan\theta$ only
Obtain a correct horizontal equation
Rearrange equation as a quadratic in $\tan\theta$, e.g. $3\tan^2\theta + 2\tan\theta - 1 = 0$
Solve for θ (usual requirements for solution of quadratic)
Obtain answer, e.g. 18.4°
Obtain second answer, e.g. 135°, and no others in the given interval [6]

14. M/J 16/P32/Q5

(i) *EITHER*: Express $\cos 4\theta$ in terms of $\cos 2\theta$ and/or $\sin 2\theta$
Use correct double angle formulae to express LHS in terms of $\sin\theta$ and/or $\cos\theta$
Obtain a correct expression in terms of $\sin\theta$ alone
Reduce correctly to the given form
OR: Use correct double angle formula to express RHS in terms of $\cos 2\theta$
Express $\cos^2 2\theta$ in terms of $\cos 4\theta$
Obtain a correct expression in terms of $\cos 4\theta$ and $\cos 2\theta$
Reduce correctly to the given form [4]

(ii) Use the identity and carry out a method for finding a root
Obtain answer 68.5°
Obtain a second answer, e.g. 291.5°
Obtain the remaining answers, e.g. 111.5° and 248.5°, and no others in the given interval [4]
[Ignore answers outside the given interval. Treat answers in radians as a misread.]

15. M/J 16/P31/Q3

Correctly restate the equation in terms of $\sin\theta$ and $\cos\theta$
Using Pythagoras obtain a horizontal equation in $\cos\theta$
Reduce the equation to a correct quadratic in $\cos\theta$, e.g. $3\cos^2\theta - \cos\theta - 2 = 0$
Solve a 3-term quadratic for $\cos\theta$
Obtain answer $\theta = 131.8°$ only
[Ignore answers outside the given interval.] [5]

16. M/J 16/P33/Q3

(i) State answer $R = 3$

Use trig formula to find
Obtain $\alpha = 41.81°$ with no errors seen [3]

(ii) Evaluate $\cos^{-1}(0.4)$ to at least 1 d.p. (66.42°to 2 d.p.)
Carry out an appropriate method to find a value of x in the given range
Obtain answer 216.5° only
[Ignore answers outside the given interval.] [3]

17. O/N 15/P32/Q3,O/N 15/P31/Q3

Use $\tan(A \pm B)$ and obtain an equation in $\tan\theta$ and $\tan\phi$
Substitute throughout for $\tan\theta$ or for $\tan\phi$
Obtain $3\tan^2\theta - \tan\theta - 4 = 0$ or $3\tan^2\phi - 5\tan\phi - 2 = 0$, or 3-term equivalent
Solve a 3-term quadratic and find an angle
Obtain answer $\theta = 135°$, $\phi = 63.4°$
Obtain answer $\theta = 53.1°$, $\phi = 161.6°$ [6]
[Treat answers in radians as a misread. Ignore answers outside the given interval.]
[SR: Two correct values of θ (or ϕ) score A1; then A1 for both correct θ, ϕ pairs.]

18. O/N 15/P33/Q6

State or imply $\sin A \times \cos 45 + \cos A \times \sin 45 = 2\sqrt{2}\cos A$
Divide by $\cos A$ to find value of $\tan A$
Obtain $\tan A = 3$
Use identity $\sec^2 B = 1 + \tan^2 B$
Solve three-term quadratic equation and find $\tan B$
Obtain $\tan B = \frac{3}{2}$ only
Substitute **numerical values** in $\frac{\tan A - \tan B}{1 + \tan A \tan B}$
Obtain $\frac{3}{11}$ [8]

19. M/J 15/P32/Q4

(i) State $R = \sqrt{13}$
Use trig formula to find α
Obtain $\alpha = 33.69°$ with no errors seen [3]

(ii) Evaluate $\sin^{-1}(1/\sqrt{13})$ to at least 1 d.p. (16.10° to 2 d.p)
Carry out an appropriate method to find a value of θ in the interval $0° < \theta < 180°$
Obtain answer $\theta = 130.2°$ and no other in the given interval [3]
[Ignore answers outside the given interval.]
[Treat answers in radians as a misread and deduct A1 from the marks for the angles.]

20. M/J 15/P33/Q3

Use correct $\tan 2A$ and $\cot A$ formulae to form an equation in $\tan x$
Obtain a correct equation in any form
Reduce equation to the form $\tan^2 x + 6\tan x - 3 = 0$, or equivalent
Solve a three term quadratic in $\tan x$ for x, **as in Q1.**
Obtain answer, e.g. 24.9° (24.896)
Obtain second answer, e.g. 98.8 (98.794) and no others in the given interval 6
[Ignore outside the given interval. Treat answers in radians as a misread.]
Radian answers 0.43452, 1.7243

21. O/N 14/P32/Q8,O/N 14/P31/Q8

(i) Use $\sin(A + B)$ formula to express $\sin 3\theta$ in terms of trig. functions of 2θ and θ
Use correct double angle formulae and Pythagoras to express $\sin 3\theta$ in terms of $\sin\theta$

Obtain a correct expression in terms of $\sin\theta$ in any form
Obtain the given identity **[4]**
[SR: Give M1 for using correct formulae to express RHS in terms of $\sin\theta$ and $\cos 2\theta$, then M1A1 for expressing in terms of $\sin\theta$ and $\sin 3\theta$ only, or in terms of $\cos\theta$, $\sin\theta$, $\cos 2\theta$ and $\sin 2\theta$, then A1 for obtaining the given identity.]

(ii) Substitute for x and obtain the given answer **[1]**

(iii) Carry out a correct method to find a value of x
Obtain answers 0.322, 0.799, −1.12 **[4]**
[Solutions with more than 3 answers can only earn a maximum of A1 + A1.]

22. O/N 14/P33/Q4

(i) Either Use $\cos(A \pm B)$ correctly at least once
State correct complete expansion
Confirm given answer $\cos\theta$ with explicit use of $\cos 60^o = \frac{1}{2}$
SR: "correct" answer from sign errors in both expansions is B1 only

Or Use correct $\cos A + \cos B$ formula
State correct result e.g. $2\cos\left(\frac{2\theta}{2}\right)\cos\left(\frac{-120}{2}\right)$
Confirm given answer $\cos\theta$ with explicit use of $\cos(\pm 60^o) = \frac{1}{2}$

(ii) State or imply $\frac{\cos 2x}{\cos x} = 3$
Obtain equation $2\cos^2 x - 3\cos x - 1 = 0$
Solve a three-term quadratic equation for $\cos x$
Obtain $\frac{1}{4}(3 - \sqrt{17})$ or exact equivalent and, finally, no other **[4]**

23. M/J 14/P32/Q3

Use $\cos(A + B)$ formula to obtain an equation in $\cos x$ and $\sin x$
Use trig formula to obtain an equation in $\tan x$ (or $\cos x$ or $\sin x$)
Obtain $\tan x = \sqrt{3} - 4$, or equivalent (or find $\cos x$ or $\sin x$)
Obtain answer $x = -66.2°$
Obtain answer $x = 113.8°$ and no others in the given interval **5**
[Ignore answers outside the given interval. Treat answers in radians as a misread (−1.16, 1.99).]
[The other solution methods are *via* $\cos x = \pm 1/\sqrt{(1 + (\sqrt{3} - 4)^2)}$ and $\sin x = \pm(\sqrt{3} - 4)/\sqrt{(1 + (\sqrt{3} - 4)^2)}$.]

24. M/J 14/P31/Q1

(i) State $\sin 2\alpha = 2\sin\alpha\cos\alpha$ and $\sec\alpha = 1/\cos\alpha$
Obtain $2\sin\alpha$ **[2]**

(ii) Use $\cos 2\beta = 2\cos^2\beta - 1$ or equivalent to produce correct equation in $\cos\beta$
Solve three-term quadratic equation for $\cos\beta$
Obtain $\cos\beta = \frac{1}{3}$ only **[3]**

25. M/J 14/P33/Q3

(i) Use $\tan(A \pm B)$ formula and obtain an equation in $\tan x$
Using $\tan 60° = \sqrt{3}$, obtain a horizontal equation in $\tan x$ in any correct form
Reduce the equation to the given form **3**

(ii) Solve the given quadratic for $\tan x$
Obtain a correct answer, e.g. $x = 21.6°$
Obtain a second answer, e.g. $x = 128.4°$, and no others **3**
[Ignore answers outside the given interval. Treat answers in radians as a misread (0.377, 2.24).]

26. O/N 13/P33/Q7

(i) Use $\sec\theta = \dfrac{1}{\cos\theta}$ and $\mathrm{cosec}\theta = \dfrac{1}{\sin\theta}$
Use $\sin 2\theta = 2\sin\theta\cos\theta$ and to form a horizontal equation in $\sin\theta$ and $\cos\theta$ or fractions with common denominators
Obtain given equation $2\sin\theta + 4\cos\theta = 3$ correctly [3]

(ii) State or imply $R = \sqrt{20}$ or 4.47 or equivalent
Use correct trigonometry to find α
Obtain 63.43 or 63.44 with no errors seen [3]

(iii) Carry out a correct method to find one value in given range
Obtain 74.4° (or 338.7°)
Carry out a correct method to find second value in given range
Obtain 338.7° (or 74.4°) and no others between 0° and 360° [4]

27. M/J 13/P32/Q7

(i) Use cos $(A + B)$ formula to express the given expression in terms of $\cos x$ and $\sin x$
Collect terms and reach $\dfrac{\cos x}{\sqrt{2}} - \dfrac{3}{\sqrt{2}}\sin x$, or equivalent
Obtain $R = 2.236$
Use trig formula to find α
Obtain $\alpha = 71.57°$ with no errors seen [5]

(ii) Evaluate $\cos^{-1}(2/2.236)$ to at least 1 d.p. (26.56° to 2 d.p., use of $R = \sqrt{5}$ gives 26.57°)
Carry out an appropriate method to find a value of x in the interval $0° < x < 360°$
Obtain answer, e.g. $x = 315°$ (315.0°)
Obtain second answer, e.g. 261.9° and no others in the given interval [4]
[Ignore answers outside the given range.]
[Treat answers in radians as a misread and deduct A1 from the answers for the angles.]
[SR: Conversion of the equation to a correct quadratic in $\sin x$, $\cos x$, or $\tan x$ earns B1, then M1 for solving a 3-term quadratic and obtaining a value of x in the given interval, and A1 + A1 for the two correct answers (candidates must reject spurious roots to earn the final A1).]

28. M/J 13/P33/Q3

Use correct tan $2A$ formula and $\cot x = 1/\tan x$ to form an equation in $\tan x$
Obtain a correct horizontal equation in any form
Solve an equation in $\tan^2 x$ for x
Obtain answer, e.g. 40.2°
Obtain second answer, e.g. 139.8°, and no other in the given interval [5]
[Ignore answers outside the given interval.]
[Treat answers in radians as a misread and deduct A1 from the marks for the angles.]
[SR: For the answer $x = 90°$ give B1 and A1 for one of the other angles.]

29. O/N 12/P32/Q3, O/N 12/P31/Q3

Attempt use of $\sin(A + B)$ and $\cos(A - B)$ formulate to obtain an equation in $\cos\theta$ and $\sin\theta$
Obtain a correct equation in any form
Use trig. formula to obtain an equation in $\tan\theta$ (or $\cos\theta$, $\sin\theta$ or $\cot\theta$)
Obtain $\tan\theta = \dfrac{\sqrt{6}-1}{1-\sqrt{2}}$, or equivalent (or find cost θ, sin θ or cot θ) [5]
Obtain answer $\theta = 105.9°$, and no others in the given interval
[Ignore answers outside the given material]

30. O/N 12/P33/Q2

(i) State or imply $R = 25$
Use correct trigonometric formula to find α [3]
Obtain 16.26° with no errors seen

(ii) Evaluate of $\sin^{-1}\dfrac{17}{R}$ (= 42.84...°)
Obtain answer 59.1° **[2]**

31. M/J 12/P32/Q4

Use trig formulae to express equation in terms of cos θ and sin θ
Use Pythagoras to obtain an equation in sin θ
Obtain 3-term quadratic $2\sin^2\theta - 2\sin\theta - 1 = 0$, or equivalent
Solve a 3-term quadratic and obtain a value of θ **[6]**
Obtain answer, e.g. 201.5°
Obtain second answer, e.g. 338.5°, and no others in the given interval
[Ignore answers outside the given interval. Treat answers in radians (3.52, 5.91) as a misread and deduct A1 from the marks for the angles.]

32. M/J 12/P32/Q6

(i) State derivative in any correct form, e.g. $3\cos x - 12\cos^2 x\sin x$
Equate derivative to zero and solve for sin 2x, or sin x or cos x
Obtain answer $x = \dfrac{1}{12}\pi$
Obtain answer $x = \dfrac{5}{12}\pi$
Obtain answer $x = \dfrac{1}{2}\pi$ and no others in the given interval **[6]**

(ii) Carry out a method for determining the nature of the relevant stationary point
Obtain a maximum at $\dfrac{1}{12}\pi$ correctly **[2]**
[Treat answers in degrees as a misread and deduct A1 from the marks for the angles.]

33. M/J 12/P33/Q6

(i) Use tan $(A + B)$ and tan $2A$ formulae to obtain an equation in tan x
Obtain a correct equation in tan x in any form
Obtain an expression of the form $a\tan^2 x = b$
Obtain the given answer **[4]**

(ii) Substitute $k = 4$ in the given expression and solve for x
Obtain answer, e.g. $x = 16.8°$
Obtain second answer, e.g. $x = 163.2°$, and no others in the given interval **[3]**
[Ignore answers outside the given interval. Treat answers in radians as a misread and deduct A1 from the marks for the angles.]

(iii) Substitute $k = 2$, show $\tan^2 x < 0$ and justify given statement correctly **[1]**

34. O/N 11/P32/Q6

(i) State or imply $R = \sqrt{10}$
Use trig formulae to find α
Obtain $\alpha = 71.57°$ with no errors seen **[3]**
[Do not allow radians in this part. If the only trig error is a sign error in $\cos(x - \alpha)$ give M1A0]

(ii) Evaluate $\cos^{-1}(2/\sqrt{10})$ correctly to at least 1 d.p. (50.7684...°) (Allow 50.7° here)
Carry out an appropriate method to find a value of 2θ in $0° < 2\theta < 180°$
Obtain an answer for θ in the given range, e.g. $\theta = 61.2°$
Use an appropriate method to find another value of 2θ in the above range
Obtain second angle, e.g. $\theta = 10.4°$, and no others in the given range **[4]**
[Ignore answers outside the given range.]
[Treat answers in radians as a misread and deduct A1 from the answers for the angles.]
[SR: The use of correct trig formulae to obtain a 3-term quadratic in tan θ, sin 2θ, cos 2θ, or tan 2θ earns M1; then A1 for a correct quadratic, M1 for obtaining a value of θ in the given range, and A1 + A1 for the two correct answers (candidates who square must reject the spurious roots to get the final A1).]

35. O/N 11/P31/Q2

EITHER: Use chain rule

obtain $\frac{dx}{dt} = 6\sin t\cos t$, or equivalent

obtain $\frac{dy}{dt} = -6\cos^2 t\sin t$, or equivalent

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Obtain final answer $\frac{dy}{dx} = -\cos t$

OR: Express y in terms of x and use chain rule

Obtain $\frac{dy}{dx} = k(2 - \frac{x}{3})^{\frac{1}{2}}$, or equivalent

Obtain $\frac{dy}{dx} = -(2 - \frac{x}{3})^{\frac{1}{2}}$, or equivalent

Express derivative in terms of t

Obtain final answer $\frac{dy}{dx} = -\cos t$ [5]

36. O/N 11/P31/Q6

(i) State or imply $R = \sqrt{10}$

Use trig formulae to find α

Obtain $\alpha = 71.57°$ with no errors seen [3]

[Do not allow radians in this part. If the only trig error is a sign error in $\cos(x - \alpha)$ give M1A0]

(ii) Evaluate $\cos^{-1}(2/\sqrt{10})$ correctly to at least 1 d.p. (50.7684…°) (Allow 50.7° here)

Carry out an appropriate method to find a value of 2θ in $0° < 2\theta < 180°$

Obtain an answer for θ in the given range, e.g. $\theta = 61.2°$

Use an appropriate method to find another value of 2θ in the above range

Obtain second angle, e.g. $\theta = 10.4°$, and no others in the given range [5]

[Ignore answers outside the given range.]

[Treat answers in radians as a misread and deduct A1 from the answers for the angles.]

[SR: The use of correct trig formulae to obtain a 3-term quadratic in $\tan\theta$, $\sin 2\theta$, $\cos 2\theta$, or $\tan 2\theta$ earns M1; then A1 for a correct quadratic, M1 for obtaining a value of θ in the given range, and A1 + A1 for the two correct answers (candidates who square must reject the spurious roots to get the final A1).]

37. O/N 11/P33/Q3

(i) State or imply $R = 17$

Use correct trigonometric formula to find α

Obtain 61.93° with no errors seen [3]

(ii) Evaluate $\cos^{-1}\frac{12}{R}$ ($= 45.099$)

Obtain answer 107.0°

Carry out correct method for second answer

Obtain answer 16.8° and no others between 0° and 360° [4]

38. M/J 11/P32/Q3

Use correct trig formula (or formulae) and obtain an equation in $\cos\theta$

Obtain $8\cos^2\theta + \cos\theta - 7 = 0$, or equivalent

Solve a 3-term quadratic in $\cos\theta$ and reach $\theta = \cos^{-1}(a)$

Obtain answer 29.0° [5]

Obtain answer 180° and no others

[Ignore answers outside the given interval. Treat answers in radians (0.505 and 3.14 or π) as a misread.]

[SR: The answer 180° found by inspection can earn B1.]

39. M/J 11/P33/Q4

(i) Use $\tan(A \pm B)$ formula correctly at least once and obtain an equation in $\tan\theta$
Obtain a correct horizontal equation in any form
Use $\tan 60° = \sqrt{3}$ throughout
Obtain the given equation correctly [4]

(ii) Set $k = 3\sqrt{3}$ and obtain $\tan^2\theta = \frac{1}{11}$
Obtain answer 16.8°
Obtain answer 163.2° [3]
[Ignore answers outside the given interval. Treat answers in radians (0.293 and 2.85) as a misread.]

40. O/N 10/P32/Q3, O/N 10/P31/Q3

Attempt use of $\cos(A + B)$ formula to obtain an equation in $\cos\theta$ and $\sin\theta$
Use trig formula to obtain an equation in $\tan\theta$ (or $\cos\theta$, $\sin\theta$ or $\cot\theta$)
Obtain $\tan\theta = 1/(4 + \sqrt{3})$ or equivalent (or find $\cos\theta$, $\sin\theta$ or $\cot\theta$)
Obtain answer $\theta = 9.9°$
Obtain $\theta = 189.9°$, and no others in the given interval [5]
[Ignore answers outside the given interval. Treat answers in radians as a misread (0.173, 3.31).]

[The other solution methods are *via* $\cos\theta = \pm(4 + \sqrt{3})/\sqrt{\left(1+\left(4+\sqrt{3}\right)^2\right)}$ or

$\sin\theta = \pm 1/\sqrt{\left(1+\left(4+\sqrt{3}\right)^2\right)}$.]

41. O/N 10/P33/Q8

(i) Obtain or imply $R = 4$
Use appropriate trigonometry to find α
Obtain $\alpha = 52.24$ or better from correct work [3]

(ii) (a) State or imply $\theta - \alpha = \cos^{-1}(-4 \div R)$
Obtain 232.2 or better [2]

(b) Attempt at least one value using $\cos^{-1}(3 \div R)$
Obtain one correct value e.g. $\pm 41.41°$
Use $\frac{1}{2}\theta - \alpha = \cos^{-1}\left(\frac{3}{R}\right)$ to find θ
Obtain 21.7 [4]

42. M/J 10/P32/Q3

(i) State or imply $\sin a = 4/5$
Use $\sin(A - B)$ formula and substitute for $\cos a$ and $\sin a$
Obtain answer $\frac{1}{10}(4\sqrt{3} - 3)$, or exact eqivalent [3]

(ii) Use $\tan 2A$ formula and substitute for $\tan a$, or use $\sin 2A$ and $\cos 2A$ formulae, substitute $\sin a$ and $\cos a$, and divide
Obtain $\tan 2a = -\frac{24}{7}$, or equivalent
Use $\tan(A + B)$ formula with $A = 2a$, $B = a$ and substitute for $\tan 2a$ and $\tan a$
Obtain $\tan 3a = -\frac{44}{117}$ [4]

43. M/J 10/P31/Q2

Use correct $\cos 2A$ formula and obtain an equation in $\sin\theta$
Obtain $4\sin^2\theta + \sin\theta - 3 = 0$, or equivalent

Make reasonable attempt to solve a 3-term quadratic in sin θ
Obtain answer 48.6°
Obtain answer 131.4° and no others in the given range
Obtain answer 270° and no others in the given range
[Treat the giving of answers in radians as a misread. Ignore answers outside the given range.] [6]

44. M/J 10/P33/Q3

Attempt to use $\tan(A \pm B)$ formula and obtain an equation in $\tan x$
Obtain 3-term quadratic $2\tan^2 x + 3\tan x - 1 = 0$, or equivalent
Solve a 3-term quadratic and find a numerical value of x
Obtain answer 15.7°
Obtain answer 119.3° and no others in the given interval
[Ignore answers outside the given interval. Treat answers in radians, 0.274 and 2.08, as a misread.] [5]

45. O/N 09/P32/Q4

Use $\tan(A \pm B)$ formula and obtain an equation in $\tan\alpha$ and $\tan\beta$
Substitute throughout for $\tan\alpha$ or for $\tan\beta$
Obtain $2\tan^2\beta + \tan\beta - 1 = 0$ or $\tan^2\alpha + \tan\alpha - 2 = 0$, or equivalent
Solve a 3-term quadratic and find an angle
Obtain answer $\alpha = 45°, \beta = 26.6°$
Obtain answer $\alpha = 116.6°, \beta = 135°$
[Treat answers given in radians as a misread. Ignore answers outside the given range.] [6]
[SR: Two correct values of α (or β) score A1; then A1 for both correct α, β pairs]

46. M/J 09/P3/Q3

(i) Use $\cot A = 1/\tan A$ or $\cos A/\sin A$ and/or $\operatorname{cosec} A = 1/\sin A$ on at least two terms
Use a correct double angle formula or the $\sin(A - B)$ formula at least once
Obtain given result 1 3

(ii) Solve $\cot\theta = 2$ for θ and obtain answer 26.6°
Obtain answer 206.6° and no others in the given range 2
[Ignore answers outside the given range. Treat answers given in radians as a misread]

47. M/J 09/P3/Q6

(i) EITHER State $\frac{dx}{dt} = -3a\cos^2 t\sin t$ or $\frac{dy}{dt} = 3a\sin^2 t\cos t$, or equivalent

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

OR State $\frac{2}{3}x^{-\frac{1}{3}}dx$ or $\frac{2}{3}y^{-\frac{1}{3}}dy$ as differentials of $x^{\frac{2}{3}}$ or $y^{\frac{2}{3}}$ respectively, or equivalent

Obtain $\frac{dy}{dx}$ in terms of t, having taken the differential of a constant to be zero M1

Obtain $\frac{dy}{dx}$ in any correct form 3

(ii) Form the equation of the tangent
Obtain the equation in any correct form
Obtain the given answer 3

(iii) State the x-coordinate of X or the y-coordinate of Y in any correct form
Obtain the given answer with no errors seen 2

48. O/N 08/P3/Q6

(i) State or imply at any stage answer $R = 13$
Use trig formula to find α
Obtain $\alpha = 67.38°$ with no errors seen [3]
[Do not allow radians in this part. If the only trig error is a sign error in $\sin(x + \alpha)$ give M1A0.]

(ii) Evaluate $\sin^{-1}\left(\frac{11}{13}\right)$ correctly to at least 1 d.p (57.79577...°)
Carry out an appropriate method to find a value of 2θ in $0° < 2\theta < 360°$
Obtain an answer for θ in the given range, e.g. $\theta = 27.4°$
Use an appropriate method to find another value of 2θ in the above range
Obtain second angle, e.g. $\theta = 175.2°$ and no others in the given range **[5]**
[Ignore answers outside the given range.]
[Treat answers in radians as a misread and deduct A1 from the answers for the angles.]
[SR: The use of correct trig formulae to obtain a 3-term quadratic in $\tan\theta$, $\sin 2\theta$, $\cos 2\theta$, or $\tan 2\theta$ earns M1; then A1 for a correct quadratic, M1 for obtaining a value of θ in the given range, and A1 + A1 for the two correct answers (candidates who square must reject the spurious roots to get the final A1).]

49. M/J 08/P3/Q4

(i) Use $\tan(A \pm B)$ formula correctly at least once to obtain an equation in $\tan\theta$
Obtain a correct horizontal equation in any form
Use correct exact values of tan 30° and tan 60° throughout **[4]**
Obtain the given equation correctly

(ii) Make reasonable attempt to solve the given quadratic in $\tan\theta$
Obtain answer $\theta = 24.7°$ **[3]**
Obtain answer $\theta = 95.3°$ and no others in the given range
[Ignore answers outside the given range.]
[Treat answers in radians as MR and deduct one mark from the marks for the angles.]

50. O/N 07/P3/Q5

(i) Use correct $\tan(A + B)$ formula to obtain an equation in $\tan x$
Use $\tan 45° = 1$ **[3]**
Obtain the given answer

(ii) Make reasonable attempt to solve the given quadratic for one value of $\tan x$
Obtain $\tan x = -1 \pm \sqrt{2}$, or equivalent in the form $(a \pm \sqrt{b})/c$ (accept 0.4, −2.4)
Obtain answer $x = 22.5°$ **[4]**
Obtain second answer $x = 112.5$ and no others in the range
[Ignore answers outside the range.]
[Treat answers in radians as a MR and deduct one mark from the marks for the angles.]

51. O/N 06/P3/Q2

EITHER: Use $\tan 2A$ formula and obtain a horizontal equation in $\tan x$
Simplify the equation to the form $3\tan^2 x = 1$, or equivalent
Obtain answer 30°
Obtain second answer 150° and no other in the range

OR: Use $\sin 2A$ and $\cos 2A$ formulae and obtain a horizontal equation in $\sin x$ or $\cos x$
Simplify the equation to $4\sin^2 x = 1$, $4\cos^2 x = 3$, or equivalent.
Obtain answer 30°
Obtain second answer 150° and no others in the range
[Ignore answers outside the given range.] **4**
[Treat answers in radians as a MR and deduct one mark from the marks for the angles.]
[Methods leading to an equation in $\cos 3x$ or $\cos 2x$, or to the equality of two tangents can also earn M1A1, and then A1 + A1 for 30° and 150° only.]
[SR: If the answer 30° is found by inspection or from a graph, and is exactly verified, award B2.
If a second answer 150° is found and verified, and no others stated, award B2].

52. M/J 06/P3/Q4

(i) State answer $R = 25$
Use trig formula to find a **3**
Obtain $a = 73.74°$

(ii) Carry out evaluation of $\cos^{-1}(15/25)$ (= 53.1301...°)
Obtain answer 126.9°
Carry out correct method for second answer
Obtain answer 20.6° and no others in the range **4**
[Ignore answers outside the given range.]

53. O/N 05/P3/Q5

State or imply that $R = 10$ or $R = -10$
Use trig formula to find α
Obtain $\alpha = 36.9°$ if $R = 10$ or $\alpha = 216.9°$ if $R = -10$, with no errors seen
Carry out evaluation of $\sin^{-1}(\frac{7}{10})$ ($\approx 44.427...°$)
Obtain answer 81.3°
Carry out correct method for second answer
Obtain answer 172.4° and no others in the range
[Ignore answers outside the given range.] [7]

54. M/J 05/P3/Q6

(i) EITHER: Express $\cos 4\theta$ in terms of $\cos 2\theta$ and/or $\sin 2\theta$
Use double angle formulae to express LHS in terms of $\cos\theta$ (and maybe $\sin\theta$)
Obtain any correct expression in terms of $\cos\theta$ alone
Reduce correctly to the given form

OR: Use double angle formula to express RHS in terms of $\cos 2\theta$
Express $\cos^2 2\theta$ in terms of $\cos 4\theta$.
Obtain any correct expression in terms of $\cos 4\theta$ and $\cos 2\theta$
Reduce correctly to the given form 4

(ii) Using the identity, carry out method for calculating one root
Obtain answer 27.2° (or 0.475 radians) or 27.3° (or 0.476 radians)
Obtain a second answer, e.g. 332.8° (or 5.81 radians)
Obtain remaining answers, e.g. 152.8° and 207.2°
(or 2.67 and 3.62 radians) and no others in range 4

55. O/N 04/P3/Q4

(i) *EITHER*: Use $\tan(A \pm B)$ formula correctly to obtain an equation in $\tan x$
State or imply the equation $\frac{1+\tan x}{1-\tan x} = \frac{2(1-\tan x)}{1+\tan x}$ or equivalent
Transform to an expanded horizontal quadratic equation in $\tan x$
Obtain given answer correctly

OR: Use $\sin(A \pm B)$ and $\cos(A \pm B)$ formulae correctly to obtain an equation in $\sin x$ and $\cos x$
Using values of sin 45°and cos 45°, or their equality, obtain an expanded horizontal equation in $\sin x$ and $\cos x$
Transform to a quadratic equation in $\tan x$
Obtain given answer correctly 4

(ii) Solve the given quadratic and calculate an angle in degrees or radians.
Obtain one answer e.g. 80.3°
Obtain second answer 9.7° and no others in the range 3
[Ignore answers outside the given range.]

56. M/J 04/P3/Q1

Show correct sketch for $0 \le x < \frac{1}{2}\pi$
Show correct sketch for $\frac{1}{2}\pi < x < \frac{3}{2}\pi$ or $\frac{3}{2}\pi < x \le 2\pi$
Show completely correct sketch 3
[SR: for a graph with $y = 0$ when $x = 0, \pi, 2\pi$ but otherwise of correct shape, award B1.]

57. O/N 03/P3/Q3

Use correct $\cos 2A$ formula, or equivalent pair of correct formulas, to obtain an equation in $\cos\theta$
Obtain 3-term quadratic $6\cos^2\theta + \cos\theta - 5 = 0$, or equivalent

Attempt to solve quadratic and reach $\theta = \cos^{-1}(a)$
Obtain answer 33.6° (or 33.5°) or 0.586 (or 0.585) radians
Obtain answer 180° or π (or 3.14) radians and no others in range
[The answer θ = 180° found by inspection can earn B1.]
[Ignore answers outside the given range.] **[5]**

58. M/J 03/P3/Q1

(i) Use trig formulae to express *LHS* in terms of sin *x* and cos *x*
Use cos 60° = sin 30° to reduce equation to given form cos *x* = *k* **[2]**

(ii) State or imply that $k = -\frac{1}{\sqrt{3}}$ (accept -0.577 or -0.58)

Obtain answer x = 125.3° only
[Answer must be in degrees; ignore answers outside the given range.]

[SR: if $k = \frac{1}{\sqrt{3}}$ is followed by *x* = 54.7°, give A0A1√.] **[2]**

59. O/N 02/P3/Q5

(i) State or imply at any stage that $R = 5$
Use trig formula to find a
Obtain answer $a = 36.87°$ **3**

(ii) *EITHER:* Carry out, or indicate need for, calculation of $\sin^{-1}\left(\frac{2}{5}\right)$

Obtain answer 60.4° (or 60.5°)
Carry out correct method for second root i.e. 180° – 23.578° + 36.870°
Obtain answer 193.3° and no others in range

OR Obtain a three-term quadratic equation in sin θ or cos θ
Solve a two- or three- term quadratic and calculate an angle.
Obtain answer 60.4° (or 60.5°)
Obtain answer 193.3° and no others in range. **4**

(iii) State greatest value is 1 **1**
[Treat work in radians as a misread, scoring a maximum of 7. The angles are 0.644, 1.06 and 3.37.]

60. M/J 02/P3/Q1

EITHER: Express LHS in terms of cos θ and sin θ or terms of tan θ
Make sufficient relevant use of double-angle formula(e)
Complete proof of the result

OR: Express RHS in terms of cos θ and sin θ or in terms of tan θ
Express RHS as the difference (or sum) of two fractions
Complete proof of the result **3**

[SR: an attempt ending with $\frac{1 + \tan^2\theta}{\tan\theta} = \cot\theta - \tan\theta$ earns M1 B0 only.]

UNIT 4

Differentiation

A-Level
Mathematics Paper 3
Topical Workbook

READ & WRITE
PUBLICATIONS
Sale Point: Shop # 25-28 Lower Ground Floor,
Haadia Haleema Centre, Ghazni Street,
Urdu Bazar, Lahore.
Tel: +92 42-35714038
Cell: +92 336-5314141
Web: www.readnwrite.org
E-mail: readandwriteoffice@gmail.com

Unit-4: Differentiation

1. M/J 18/P32/Q5

The equation of a curve is $x^2(x + 3y) - y^3 = 3$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{x^2 + 2xy}{y^2 - x^2}$. [4]

(ii) Hence find the exact coordinates of the two points on the curve at which the gradient of the normal is 1. [4]

2. M/J 18/P31/Q3

A curve has equation $y = \dfrac{e^{3x}}{\tan \frac{1}{2}x}$. Find the x-coordinates of the stationary points of the curve in the interval $0 < x < \pi$. Give your answers correct to 3 decimal places. [6]

3. M/J 18/P33/Q8

The equation of a curve is $2x^3 - y^3 - 3xy^2 = 2a^3$, where a is a non-zero constant.

(i) Show that $\dfrac{dy}{dx} = \dfrac{2x^2 - y^2}{y^2 + 2xy}$. [4]

(ii) Find the coordinates of the two points on the curve at which the tangent is parallel to the y-axis. [5]

4. O/N 17/P32/Q4

The curve with equation $y = \dfrac{2 - \sin x}{\cos x}$ has one stationary point in the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

(i) Find the exact coordinates of this point. [5]

(ii) Determine whether this point is a maximum or a minimum point. [2]

5. O/N 17/P32/Q6

The equation of a curve is $x^3y - 3xy^3 = 2a^4$, where a is a non-zero constant.

(i) Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - 3y^3}{9xy^2 - x^3}$. [4]

(ii) Hence show that there are only two points on the curve at which the tangent is parallel to the x-axis and find the coordinates of these points. [4]

6. O/N 17/P31/Q5 , O/N 17/P33/Q5

The equation of a curve is $2x^4 + xy^3 + y^4 = 10$.

(i) Show that $\dfrac{dy}{dx} = -\dfrac{8x^3 + y^3}{3xy^2 + 4y^3}$. [4]

(ii) Hence show that there are two points on the curve at which the tangent is parallel to the x-axis and find the coordinates of these points. [4]

7. M/J 17/P32/Q4

The parametric equations of a curve are

$$x = t^2 + 1, \quad y = 4t + \ln(2t - 1).$$

(i) Express $\dfrac{dy}{dx}$ in terms of t. [3]

(ii) Find the equation of the normal to the curve at the point where $t = 1$. Give your answer in the form $ax + by + c = 0$. [3]

8. M/J 17/P31/Q4

The parametric equations of a curve are

$$x = \ln \cos \theta, \quad y = 3\theta - \tan \theta,$$

where $0 \leqslant \theta < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of $\tan\theta$. [5]

(ii) Find the exact y-coordinate of the point on the curve at which the gradient of the normal is equal to 1. [3]

9. M/J 17/P33/Q5

A curve has equation $y = \frac{2}{3}\ln(1 + 3\cos^2 x)$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of $\tan x$. [4]

(ii) Hence find the x-coordinate of the point on the curve where the gradient is -1. Give your answer correct to 3 significant figures. [2]

10. O/N 16/P32/Q4, O/N 16/P31/Q3

The equation of a curve is $xy(x - 6y) = 9a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of this point. [7]

11. M/J 16/P32/Q4

The curve with equation $y = \frac{(\ln x)^2}{x}$ has two stationary points. Find the exact values of the coordinates of these points. [6]

12. M/J 16/P31/Q4

The variables x and y satisfy the differential equation

$$x\frac{dy}{dx} = y(1 - 2x^2),$$

and it is given that $y = 2$ when $x = 1$. Solve the differential equation and obtain an expression for y in terms of x in a form not involving logarithms. [6]

13. M/J 16/P31/Q7

The equation of a curve is $x^3 - 3x^2y + y^3 = 3$.

(i) Show that $\frac{dy}{dx} = \frac{x^2 - 2xy}{x^2 - y^2}$. [4]

(ii) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis. [5]

14. M/J 16/P33/Q4

The parametric equations of a curve are

$$x = t + \cos t, \qquad y = \ln(1 + \sin t),$$

where $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$.

(i) Show that $\frac{dy}{dx} = \sec t$. [5]

(ii) Hence find the x-coordinates of the points on the curve at which the gradient is equal to 3. Give your answers correct to 3 significant figures. [3]

15. O/N 15/P32/Q5 , O/N 15/P31/Q5

The equation of a curve is $y = e^{-2x}\tan x$, for $0 \leqslant x < \frac{1}{2}\pi$.

(i) Obtain an expression for $\frac{dy}{dx}$ and show that it can be written in the form $e^{-2x}(a + b\tan x)^2$, where a and b are constants. [5]

(ii) Explain why the gradient of the curve is never negative. [1]

(iii) Find the value of x for which the gradient is least. [1]

16. O/N 15/P33/Q3

A curve has equation

$$y = \frac{2 - \tan x}{1 + \tan x}.$$

Find the equation of the tangent to the curve at the point for which $x = \frac{1}{4}\pi$, giving the answer in the form $y = mx + c$ where c is correct to 3 significant figures. [6]

17. M/J 15/P32/Q3

A curve has equation $y = \cos x \cos 2x$. Find the x-coordinate of the stationary point on the curve in the interval $0 < x < \frac{1}{2}\pi$, giving your answer correct to 3 significant figures. [6]

18. M/J 15/P31/Q4

The equation of a curve is

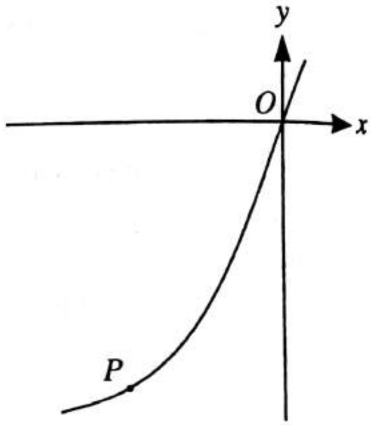
$$y = 3\cos 2x + 7\sin x + 2.$$

Find the x-coordinates of the stationary points in the interval $0 \leqslant x \leqslant \pi$. Give each answer correct to 3 significant figures. [7]

19. M/J 15/P31/Q10

The diagram shows part of the curve with parametric equations

$$x = 2\ln(t + 2), \qquad y = t^3 + 2t + 3.$$



(i) Find the gradient of the curve at the origin. [5]

(ii) At the point P on the curve, the value of the parameter is p. It is given that the gradient of the curve at P is $\frac{1}{2}$.

(a) Show that $p = \dfrac{1}{3p^2 + 2} - 2$. [1]

(b) By first using an iterative formula based on the equation in part (a), determine the coordinates of the point P. Give the result of each iteration to 5 decimal places and each coordinate of P correct to 2 decimal places. [4]

20. M/J 15/P33/Q4

The curve with equation $y = \dfrac{e^{2x}}{4 + e^{3x}}$ has one stationary point. Find the exact values of the coordinates of this point. [6]

21. M/J 15/P33/Q5

The parametric equations of a curve are

$$x = a\cos^4 t, \quad y = a\sin^4 t,$$

where a is a positive constant.

(i) Express $\dfrac{dy}{dx}$ in terms of t. [3]

(ii) Show that the equation of the tangent to the curve at the point with parameter t is

$$x\sin^2 t + y\cos^2 t = a\sin^2 t\cos^2 t.$$ [3]

(iii) Hence show that if the tangent meets the x-axis at P and the y-axis at Q, then

$$OP + OQ = a,$$

where O is the origin. [2]

22. O/N 14/P32/Q4 , O/N 14/P31/Q4

The parametric equations of a curve are

$$x = \frac{1}{\cos^3 t}, \quad y = \tan^3 t,$$

where $0 \leqslant t < \frac{1}{2}\pi$.

(i) Show that $\dfrac{dy}{dx} = \sin t$. [4]

(ii) Hence show that the equation of the tangent to the curve at the point with parameter t is $y = x\sin t - \tan t$. [3]

23. O/N 14/P33/Q2

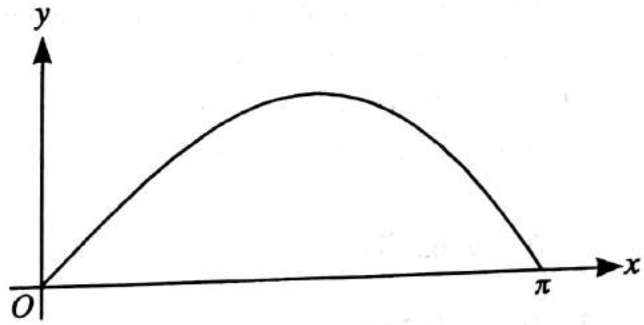
A curve is defined for $0 < \theta < \frac{1}{2}\pi$ by the parametric equations

$$x = \tan\theta, \quad y = 2\cos^2\theta\sin\theta.$$

Show that $\dfrac{dy}{dx} = 6\cos^5\theta - 4\cos^3\theta$. [5]

24. M/J 14/P32/Q8

The diagram shows the curve $y = x\cos\frac{1}{2}x$ for $0 \leqslant x \leqslant \pi$.



(i) Find $\dfrac{dy}{dx}$ and show that $4\dfrac{d^2y}{dx^2} + y + 4\sin\frac{1}{2}x = 0$. [5]

(ii) Find the exact value of the area of the region enclosed by this part of the curve and the x-axis. [5]

25. O/N 13/P32/Q1

The equation of a curve is $y = \dfrac{1+x}{1+2x}$ for $x > -\frac{1}{2}$. Show that the gradient of the curve is always negative. [3]

26. O/N 13/P32/Q4

The parametric equations of a curve are

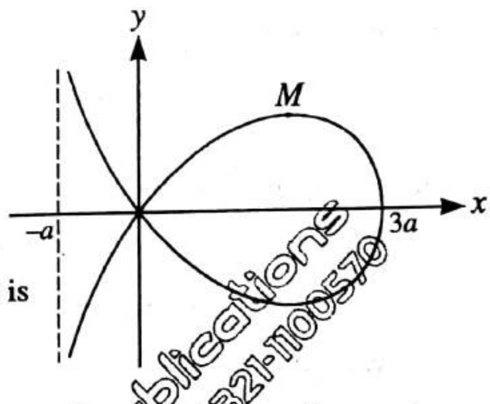
$$x = e^{-t}\cos t, \quad y = e^{-t}\sin t.$$

Show that $\dfrac{dy}{dx} = \tan\left(t - \frac{1}{4}\pi\right)$. [6]

27. M/J 13/P32/Q5

The diagram shows the curve with equation

$$x^3 + xy^2 + ay^2 - 3ax^2 = 0,$$



where a is a positive constant. The maximum point on the curve is M. Find the x-coordinate of M in terms of a. [6]

28. M/J 13/P31/Q5

For each of the following curves, find the gradient at the point where the curve crosses the y-axis:

(i) $y = \dfrac{1+x^2}{1+e^{2x}}$; [3]

(ii) $2x^3 + 5xy + y^3 = 8$. [4]

29. O/N 12/P32/Q7 , O/N 12/P31/Q7

The equation of a curve is $\ln(xy) - y^3 = 1$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{y}{x(3y^3 - 1)}$. [4]

(ii) Find the coordinates of the point where the tangent to the curve is parallel to the y-axis, giving each coordinate correct to 3 significant figures. [4]

30. O/N 12/P33/Q3

The parametric equations of a curve are

$$x = \frac{4t}{2t+3}, \quad y = 2\ln(2t+3).$$

(i) Express $\frac{dy}{dx}$ in terms of t, simplifying your answer. [4]

(ii) Find the gradient of the curve at the point for which $x = 1$. [2]

31. M/J 12/P31/Q6

The equation of a curve is $3x^2 - 4xy + y^2 = 45$.

(i) Find the gradient of the curve at the point $(2, -3)$. [4]

(ii) Show that there are no points on the curve at which the gradient is 1. [3]

32. M/J 12/P33/Q3

The parametric equations of a curve are

$$x = \sin 2\theta - \theta, \quad y = \cos 2\theta + 2\sin\theta.$$

Show that $\frac{dy}{dx} = \frac{2\cos\theta}{1 + 2\sin\theta}$. [5]

33. M/J 12/P33/Q4

The curve with equation $y = \frac{e^{2x}}{x^3}$ has one stationary point.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

34. O/N 11/P32/Q2

The parametric equations of a curve are

$$x = 3(1 + \sin^2 t), \quad y = 2\cos^3 t.$$

Find $\frac{dy}{dx}$ in terms of t, simplifying your answer as far as possible. [5]

35. O/N 11/P33/Q2

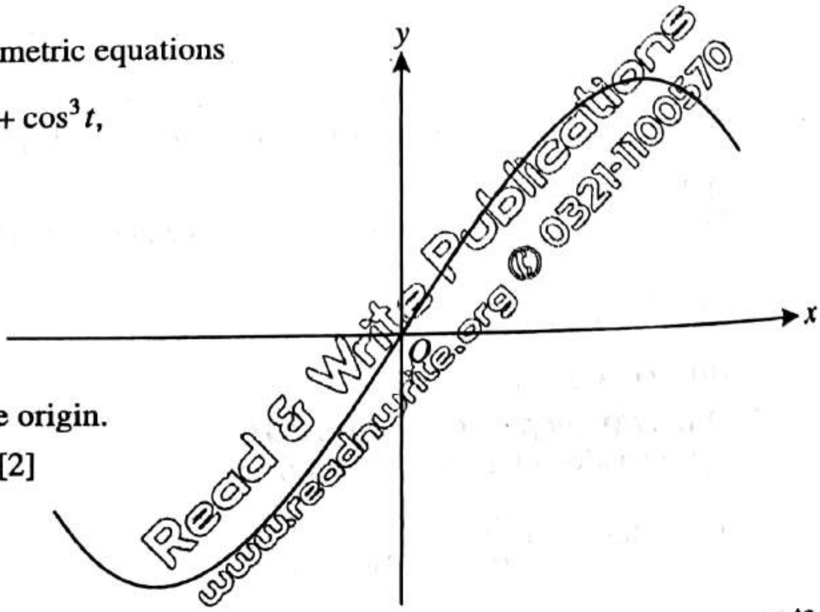
The equation of a curve is $y = \frac{e^{2x}}{1 + e^{2x}}$. Show that the gradient of the curve at the point for which $x = \ln 3$ is $\frac{9}{50}$. [4]

36. O/N 11/P33/Q8

The diagram shows the curve with parametric equations

$$x = \sin t + \cos t, \quad y = \sin^3 t + \cos^3 t,$$

for $\frac{1}{4}\pi < t < \frac{5}{4}\pi$.



(i) Show that $\frac{dy}{dx} = -3\sin t\cos t$. [3]

(ii) Find the gradient of the curve at the origin. [2]

(iii) Find the values of t for which the gradient of the curve is 1, giving your answers correct to 2 significant figures. [4]

37. M/J 11/P32/Q5

The parametric equations of a curve are

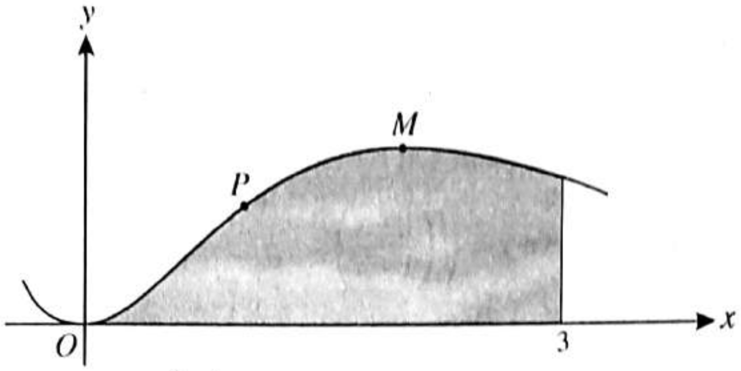
$$x = \ln(\tan t), \quad y = \sin^2 t,$$

where $0 < t < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of t. **[4]**

(ii) Find the equation of the tangent to the curve at the point where $x = 0$. **[3]**

38. M/J 11/P32/Q10



The diagram shows the curve $y = x^2 e^{-x}$.

(i) Show that the area of the shaded region bounded by the curve, the x-axis and the line $x = 3$ is equal to $2 - \frac{17}{e^3}$. **[5]**

(ii) Find the x-coordinate of the maximum point M on the curve. **[4]**

(iii) Find the x-coordinate of the point P at which the tangent to the curve passes through the origin. **[2]**

39. M/J 11/P31/Q2

Find $\frac{dy}{dx}$ in each of the following cases:

(i) $y = \ln(1 + \sin 2x)$, **[2]**

(ii) $y = \frac{\tan x}{x}$. **[2]**

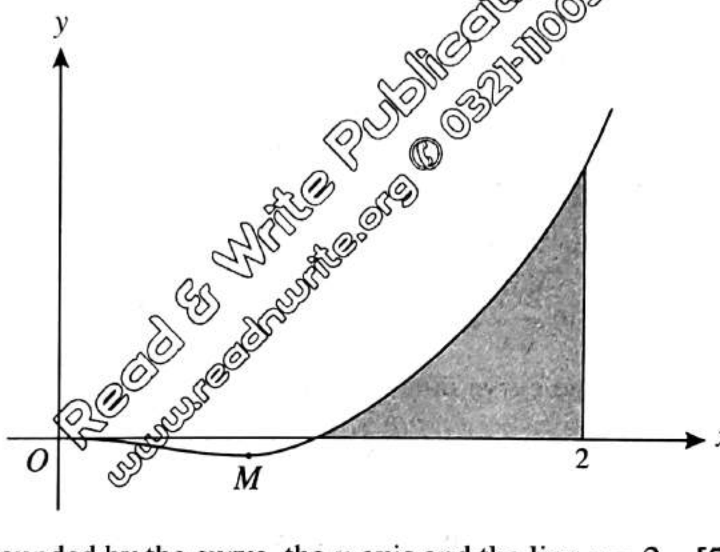
40. M/J 11/P33/Q2

The curve $y = \frac{\ln x}{x^3}$ has one stationary point. Find the x-coordinate of this point. **[4]**

41. O/N 10/P32/Q9 , O/N 10/P31/Q9

The diagram shows the curve $y = x^3 \ln x$ and its minimum point M.

(i) Find the exact coordinates of M. [5]



(ii) Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 2$. **[5]**

42. O/N 10/P33/Q2

The parametric equations of a curve are

$$x = \frac{t}{2t+3}, \qquad y = e^{-2t}.$$

Find the gradient of the curve at the point for which $t = 0$. [5]

43. M/J 10/P32/Q6

The equation of a curve is

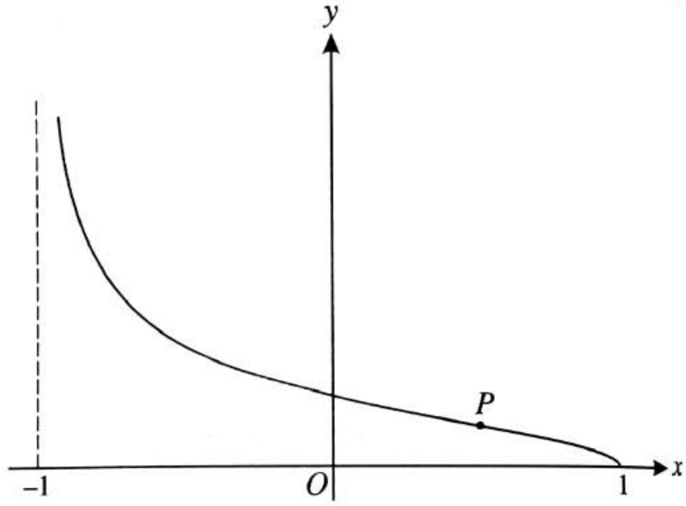
$$x \ln y = 2x + 1.$$

(i) Show that $\frac{dy}{dx} = -\frac{y}{x^2}$. [4]

(ii) Find the equation of the tangent to the curve at the point where $y = 1$, giving your answer in the form $ax + by + c = 0$. [4]

44. M/J 10/P31/Q9

The diagram shows the curve $y = \sqrt{\left(\frac{1-x}{1+x}\right)}$.



(i) By first differentiating $\frac{1-x}{1+x}$, obtain an expression for $\frac{dy}{dx}$ in terms of x. Hence show that the gradient of the normal to the curve at the point (x, y) is $(1+x)\sqrt{(1-x^2)}$. [5]

(ii) The gradient of the normal to the curve has its maximum value at the point P shown in the diagram. Find, by differentiation, the x-coordinate of P. [4]

45. O/N 09/P32/Q3

The equation of a curve is $x^3 - x^2y - y^3 = 3$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [4]

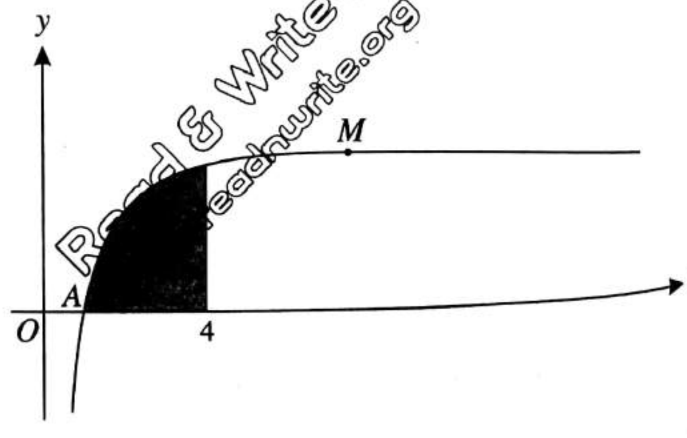
(ii) Find the equation of the tangent to the curve at the point (2, 1), giving your answer in the form $ax + by + c = 0$. [2]

46. O/N 09/P31/Q4

A curve has equation $y = e^{-3x} \tan x$. Find the x-coordinates of the stationary points on the curve in the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. Give your answers correct to 3 decimal places. [6]

47. O/N 09/P31/Q9

The diagram shows the curve $y = \frac{\ln x}{\sqrt{x}}$ and its maximum point M. The curve cuts the x-axis at the point A.



(i) State the coordinates of A. [1]

(ii) Find the exact value of the x-coordinate of M. [4]

(iii) Using integration by parts, show that the area of the shaded region bounded by the curve, the x-axis and the line $x = 4$ is equal to $8 \ln 2 - 4$. **[5]**

48. O/N 08/P03/Q3

The curve $y = \dfrac{e^x}{\cos x}$, for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, has one stationary point. Find the x-coordinate of this point. [5]

49. O/N 08/P03/Q4

The parametric equations of a curve are

$$x = a(2\theta - \sin 2\theta), \qquad y = a(1 - \cos 2\theta).$$

Show that $\dfrac{dy}{dx} = \cot \theta$. [5]

50. M/J 08/P03/Q6

The equation of a curve is $xy(x + y) = 2a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of this point. [8]

51. O/N 07/P03/Q4

The curve with equation $y = e^{-x} \sin x$ has one stationary point for which $0 \leqslant x \leqslant \pi$.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

52. M/J 07/P03/Q3

The equation of a curve is $y = x \sin 2x$, where x is in radians. Find the equation of the tangent to the curve at the point where $x = \frac{1}{4}\pi$. [4]

53. O/N 06/P03/Q3

The curve with equation $y = 6e^x - e^{3x}$ has one stationary point.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

54. O/N 06/P03/Q6

The equation of a curve is $x^3 + 2y^3 = 3xy$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{y - x^2}{2y^2 - x}$. [4]

(ii) Find the coordinates of the point, other than the origin, where the curve has a tangent which is parallel to the x-axis. [5]

55. M/J 06/P03/Q3

The parametric equations of a curve are

$$x = 2\theta + \sin 2\theta, \quad y = 1 - \cos 2\theta.$$

Show that $\dfrac{dy}{dx} = \tan \theta$. [5]

56. O/N 05/P03/Q3

The equation of a curve is $y = x + \cos 2x$. Find the x-coordinates of the stationary points of the curve for which $0 \leqslant x \leqslant \pi$, and determine the nature of each of these stationary points. [7]

57. M/J 04/P03/Q3

Find the gradient of the curve with equation

$$2x^2 - 4xy + 3y^2 = 3,$$

at the point (2, 1). [4]

58. O/N 03/P03/Q4

The equation of a curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a},$$

where a is a positive constant.

(i) Express $\frac{dy}{dx}$ in terms of x and y. [3]

(ii) The straight line with equation $y = x$ intersects the curve at the point P. Find the equation of the tangent to the curve at P. [3]

59. O/N 02/P03/Q4

The curve $y = e^{x} + 4e^{-2x}$ has one stationary point.

(i) Find the x–coordinate of this point. [4]

(ii) Determine whether the stationary point is a maximum or a minimum point. [2]

60. M/J 02/P03/Q5

The equation of a curve is $y = 2\cos x + \sin 2x$. Find the x-coordinates of the stationary points on the curve for which $0 < x < \pi$, and determine the nature of each of these stationary points. [7]

Answers Section

1. M/J 18/P32/Q5

(i) State or imply $3y^2\frac{dy}{dx}$ as derivative of y^3 **4**

State or imply $6xy+3x^2\frac{dy}{dx}$ as derivative of $3x^2y$

OR State or imply $2x(x+3y)+x^2\left(1+3\frac{dy}{dx}\right)$ as derivative of $x^2(x+3y)$

Equate derivative of the LHS to zero and solve for $\frac{dy}{dx}$

Obtain the given answer

(ii) Equate derivative to -1 and solve for y **4**

Use their $y=-2x$ or equivalent to obtain an equation in x or y

Obtain answer $(1,-2)$

Obtain answer $(\sqrt[3]{3}, 0)$

2. M/J 18/P31/Q3

Use quotient or product rule **6**

Obtain correct derivative in any form

Equate derivative to zero and obtain a quadratic in $\tan\frac{1}{2}x$ or an equation of the form $a\sin x=b$

Solve for x

Obtain answer 0.340

Obtain second answer 2.802 and no other in the given interval

3. M/J 18/P33/Q8

(i) State or imply $3y^2\frac{dy}{dx}$ as derivative of y^3 **4**

State or imply $3y^2+6xy\frac{dy}{dx}$ as derivative of $3xy^2$

Equate derivative of LHS to zero and solve for $\frac{dy}{dx}$

Obtain the given answer

(ii) Equate denominator to zero and solve for y **5**

Obtain $y=0$ and $x=a$

Obtain $y=\alpha x$ and substitute in curve equation to find x or to find y

Obtain $x=-a$

Obtain $y=2a$

4. O/N 17/P32/Q4

(i) Use correct product or quotient rule or rewrite as $2\sec x-\tan x$ and differentiate **5**

Obtain correct derivative in any form

Equate the derivative to zero and solve for x

Obtain $x=\frac{1}{6}\pi$

Obtain $y=\sqrt{3}$

(ii) Carry out an appropriate method for determining the nature of a stationary point **2**

Show the point is a minimum point with no errors seen

5. O/N 17/P32/Q6

(i) State or imply $3x^2y + x^3\frac{dy}{dx}$ as derivative of x^3y **4**

State or imply $9xy^2\frac{dy}{dx} + 3y^3$ as derivative of $3xy^3$

Equate derivative of the LHS to zero and solve for $\frac{dy}{dx}$

Obtain the given answer

(ii) Equate numerator to zero and use $x = -y$ to obtain an equation in x or in y **4**

Obtain answer $x = a$ and $y = -a$

Obtain answer $x = -a$ and $y = a$

Consider and reject $y = 0$ and $x = y$ as possibilities

6. O/N 17/P31/Q5 , O/N 17/P33/Q5

(i) State or imply $y^3 + 3xy^2\frac{dy}{dx}$ as derivative of xy^3 **4**

State or imply $4y^3\frac{dy}{dx}$ as derivative of y^4

Equate derivative of the LHS to zero and solve for $\frac{dy}{dx}$

Obtain the given answer

(ii) Equate numerator to zero **4**

Obtain $y = -2x$, or equivalent

Obtain an equation in x or y

Obtain final answer $x = -1, y = 2$ and $x = 1, y = -2$

7. M/J 17/P32/Q4

(i) State $\frac{dy}{dt} = 4 + \frac{2}{2t-1}$ **3**

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Obtain answer $\frac{dy}{dx} = \frac{8t-2}{2t(2t-1)}$, or equivalent e.g. $\frac{2}{t} + \frac{2}{4t^2 - 2t}$

(ii) Use correct method to find the gradient of the normal at $t = 1$ **3**

Use a correct method to form an equation for the normal at $t = 1$

Obtain final answer $x + 3y - 14 = 0$, or horizontal equivalent

8. M/J 17/P31/Q4

(i) Use chain rule to differentiate x $\left(\frac{dx}{d\theta} = -\frac{\sin\theta}{\cos\theta}\right)$ **5**

State $\frac{dy}{d\theta} = 3 - \sec^2\theta$

Use $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

Obtain correct $\frac{dy}{dx}$ in any form e.g. $\frac{3 - \sec^2\theta}{-\tan\theta}$

Obtain $\frac{dy}{dx} = \frac{\tan^2\theta - 2}{\tan\theta}$, or equivalent

(ii) Equate gradient to -1 and obtain an equation in $\tan\theta$ **3**

Solve a 3 term quadratic $\left(\tan^2\theta + \tan\theta - 2 = 0\right)$ in $\tan\theta$

Obtain $\theta = \frac{\pi}{4}$ and $y = \frac{3\pi}{4} - 1$ only

9. M/J 17/P33/Q5

(i) Use the chain rule **4**

Obtain correct derivative in any form

Use correct trigonometry to express derivative in terms of $\tan x$

Obtain $\frac{dy}{dx} = -\frac{4\tan x}{4+\tan^2 x}$, or equivalent

(ii) Equate derivative to -1 and solve a 3–term quadratic for $\tan x$ **2**

Obtain answer x=1.11 and no other in the given interval

10. O/N 16/P32/Q4, O/N 16/P31/Q3

EITHER: *EITHER*: State $2xy + x^2\frac{dy}{dx}$, or equivalent, as derivative of x^2y

State $6y^2 + 12xy\frac{dy}{dx}$, or equivalent, as derivative of $6xy^2$

OR: Differentiating LHS using correct product rule, state term $xy(1-6\frac{dy}{dx})$, or equivalent

State term $(y + x\frac{dy}{dx})(x-6y)$, or equivalent

Equate attempted derivative of LHS to zero and set $\frac{dy}{dx}$ equal to zero

Obtain a horizontal equation, e.g. $6y^2 - 2xy = 0$ (from correct work only)

Explicitly reject $y = 0$ as a possibility $py^2 - qxy = 0$

Obtain an equation in x or y

Obtain answer $(-3a, -a)$

OR: Rearrange to $y = \frac{9a^3}{x(x-6y)}$ and use correct quotient rule to obtain $-\frac{9a^3}{x^2(x-6y)^2}\times$.....

State term $(x-6y) + x(1-6y')$, or equivalent

Justify division by $x(x-6y)$

Set $\frac{dy}{dx}$ equal to zero

Obtain a horizontal equation, e.g. $6y^2 - 2xy = 0$ (from correct work only)

Obtain an equation in x or y

Obtain answer $(-3a, -a)$

11. M/J 16/P32/Q4

State or imply derivative of $(\ln x)^2$ is $\frac{2\ln x}{x}$ **[6]**

Use correct quotient or product rule

Obtain correct derivative in any form, e.g. $\frac{2\ln x}{x^2} - \frac{(\ln x)^2}{x^2}$

Equate derivative (or its numerator) to zero and solve for $\ln x$

Obtain the point (1, 0) with no errors seen

Obtain the point $(e^2, 4e^{-2})$

12. M/J 16/P31/Q4

Separate variables and attempt integration of at least one side **[6]**

Obtain term $\ln y$

Obtain terms $\ln x - x^2$

Use $x = 1$ and $y = 2$ to evaluate a constant, or as limits

Obtain correct solution in any form, e.g. $\ln y = \ln x - x^2 + \ln 2 + 1$

Obtain correct expression for y, free of logarithms, i.e. $y = 2x\exp(1-x^2)$

13. M/J 16/P31/Q7

(i) State or imply $6xy+3x^2\frac{dy}{dx}$ as derivative of $3x^2y$ [4]

State $3y^2\frac{dy}{dx}$ as derivative of y^3

Equate attempted derivative of the LHS to zero and solve for $\frac{dy}{dx}$

Obtain the given answer

(ii) Equate numerator to zero [5]

Obtain $x = 2y$, or equivalent

Obtain an equation in x or y

Obtain the point $(-2, -1)$

State the point $(0, 1.44)$

14. M/J 16/P33/Q4

(i) State $\frac{dx}{dt}=1-\sin t$ [5]

Use chain rule to find the derivative of y

Obtain $\frac{dy}{dt}=\frac{\cos t}{1+\sin t}$, or equivalent

Use $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$

Obtain the given answer correctly

(ii) State or imply $t=\cos^{-1}(\frac{1}{3})$ [3]

Obtain answers $x = 1.56$ and $x = -0.898$

15. O/N 15/P32/Q5 , O/N 15/P31/Q5

(i) State or imply that the derivative of e^{-2x} is $-2e^{-2x}$

Use product or quotient rule [5]

Obtain correct derivative in any form

Use Pythagoras

Justify the given form

(ii) Fully justify the given statement [1]

(iii) State answer $x=\frac{1}{4}\pi$ [1]

16. O/N 15/P33/Q3

Use correct quotient rule or equivalent to find first derivative

Obtain $\frac{-(1+\tan x)\sec^2 x-\sec^2 x(2-\tan x)}{(1+\tan x)^2}$ or equivalent

Substitute $x=\frac{1}{4}\pi$ to find gradient

Obtain $-\frac{3}{2}$

Form equation of tangent at $x=\frac{1}{4}\pi$

Obtain $y=-\frac{3}{2}x+1.68$ or equivalent [6]

17. M/J 15/P32/Q3

EITHER: Use correct product rule

Obtain correct derivative in any form, e.g. $-\sin x\cos 2x-2\cos x\sin 2x$

Use the correct double angle formulae to express derivative in $\cos x$ and $\sin x$, or $\cos 2x$ and $\sin x$

*OR*1: Use correct double angle formula to express y in terms of $\cos x$ and attempt differentiation

Use chain rule correctly

Obtain correct derivative in any form, e.g. $-6\cos^2 x\sin x+\sin x$

*OR*2: Use correct factor formula and attempt differentiation

Obtain correct derivative in any form, e.g. $-\frac{3}{2}\sin 3x-\frac{1}{2}\sin x$

Use correct trig formulae to express derivative in terms of $\cos x$ and $\sin x$, or $\sin x$
Equate derivative to zero and obtain an equation in one trig function
Obtain $6\cos^2 x = 1$, $6\sin^2 x = 5$, $\tan^2 x = 5$ or $3\cos 2x = -2$
Obtain answer $x = 1.15$ (or $65.9°$) and no other in the given interval **[6]**
[Ignore answers outside the given interval.]
[SR: Solution attempts following the *EITHER* scheme for the first two marks can earn the second and third method marks as follows:
Equate derivative to zero and obtain an equation in $\tan 2x$ and $\tan x$
Use correct double angle formula to obtain an equation in $\tan x$

18. M/J 15/P31/Q4

Differentiate to obtain form $a\sin 2x + b\cos x$ **[7]**
Obtain correct $-6\sin 2x + 7\cos x$
Use identity $\sin 2x = 2\sin x\cos x$
Solve equation of form $c\sin x\cos x + d\cos x = 0$ to find at least one value of x
Obtain 0.623
Obtain 2.52
Obtain 1.57 or $\frac{1}{2}\pi$ from equation of form $c\sin x\cos x + d\cos x = 0$
Treat answers in degrees as MR – 1 situation

19. M/J 15/P31/Q10

(i) Obtain $\frac{dx}{dt} = \frac{2}{t+2}$ and $\frac{dy}{dt} = 3t^2 + 2$ **[5]**
Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$
Obtain $\frac{dy}{dx} = \frac{1}{2}(3t^2+2)(t+2)$
Identify value of t at the origin as -1
Substitute to obtain $\frac{5}{2}$ as gradient at the origin

(ii) **(a)** Equate derivative to $\frac{1}{2}$ and confirm $p = \frac{1}{3p^2+2} - 2$ **[1]**

(b) Use the iterative formula correctly at least once **[4]**
Obtain value $p = -1.924$ or better $(-1.92367\ldots)$
Show sufficient iterations to justify accuracy or show a sign change in appropriate interval
Obtain coordinates $(-5.15, -7.97)$

20. M/J 15/P33/Q4

Use correct quotient or product rule
Obtain correct derivative in any form
Equate derivative to zero and obtain a horizontal equation
Carry out complete method for solving an equation of the form $ae^{3x} = b$, or $ae^{3x} = be^{2x}$
Obtain $x = \ln 2$, or exact equivalent
Obtain $y = \frac{1}{3}$, or exact equivalent **6**

21. M/J 15/P33/Q5

(i) State $\frac{dx}{dt} = -4a\cos^3 t\sin t$, or $\frac{dy}{dt} = 4a\sin^3 t\cos t$ **3**
Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Obtain correct expression for $\frac{dy}{dx}$ in a simplified form

(ii) Form the equation of the tangent 3
Obtain a correct equation in any form
Obtain the given answer

(iii) State the x-coordinate of P or the y-coordinate of Q in any form 2
Obtain the given result correctly

22. O/N 14/P32/Q4 , O/N 14/P31/Q4

(i) Use chain rule correctly at least once [4]
Obtain either $\frac{dx}{dt}=\frac{3\sin t}{\cos^4 t}$ or $\frac{dy}{dt}=3\tan^2 t\sec^2 t$, or equivalent
Use $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$
Obtain the given answer

(ii) State a correct equation for the tangent in any form [3]
Use Pythagoras
Obtain the given answer

23. O/N 14/P33/Q2

Use correct product rule or correct chain rule to differentiate y [5]
Use $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$
Obtain $\frac{-4\cos\theta\sin^2\theta+2\cos^3\theta}{\sec^2\theta}$ or equivalent
Express $\frac{dy}{dx}$ in terms of $\cos\theta$
Confirm given answer $6\cos^5\theta-4\cos^3\theta$ legitimately

24. M/J 14/P32/Q8

(i) Use product rule 5
Obtain derivative in any correct form
Differentiate first derivative using the product rule
Obtain second derivative in any correct form, e.g. $-\frac{1}{2}\sin\frac{1}{2}x-\frac{1}{4}x\cos\frac{1}{2}x-\frac{1}{2}\sin\frac{1}{2}x$
Verify the given statement

(ii) Integrate and reach $kx\sin\frac{1}{2}x+l\int\sin\frac{1}{2}x\,dx$ 5
Obtain $2x\sin\frac{1}{2}x-2\int\sin\frac{1}{2}x\,dx$, or equivalent
Obtain indefinite integral $2x\sin\frac{1}{2}x+4\cos\frac{1}{2}x$
Use correct limits $x=0$, $x=\pi$ correctly
Obtain answer $2\pi-4$, or exact equivalent

25. O/N 13/P32/Q1

Use correct quotient or product rule
Obtain correct derivative in any form
Justify the given statement [3]

26. O/N 13/P32/Q4

Use correct product or quotient rule at least once
Obtain $\frac{dx}{dt}=e^{-t}\sin t-e^{-t}\cos t$ or $\frac{dy}{dt}=e^{-t}\cos t-e^{-t}\sin t$, or equivalent
Use $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$

Obtain $\frac{dy}{dx}=\frac{\sin t-\cos t}{\sin t+\cos t}$, or equivalent

EITHER: Express $\frac{dy}{dx}$ in terms of $\tan t$ only

Show expression is identical to $\tan\left(t-\frac{1}{4}\pi\right)$

OR: Express $\tan\left(t-\frac{1}{4}\pi\right)$ in terms of $\tan t$

Show expression is identical to $\frac{dy}{dx}$ **[6]**

27. M/J 13/P32/Q5

EITHER: State $2ay\frac{dy}{dx}$ as derivative of ay^2

State $y^2+2xy\frac{dy}{dx}$ as derivative of xy^2

Equate derivative of LHS to zero and set $\frac{dy}{dx}$ equal to zero

Obtain $3x^2+y^2-6ax=0$, or horizontal equivalent

Eliminate y and obtain an equation in x

Solve for x and obtain answer $x=\sqrt{3}a$

OR1: Rearrange equation in the form $y^2=\frac{3ax^2-x^3}{x+a}$ and attempt differentiation of one side

Use correct quotient or product rule to differentiate RHS

Obtain correct derivative of RHS in any form

Set $\frac{dy}{dx}$ equal to zero and obtain an equation in x

Obtain a correct horizontal equation free of surds

Solve for x and obtain answer $x=\sqrt{3}a$

OR2: Rearrange equation in the form $y=\left(\frac{3ax^2-x^3}{x+a}\right)^{\frac{1}{2}}$ and differentiation of RHS

Use correct quotient or product rule and chain rule

Obtain correct derivative in any form

Equate derivative to zero and obtain an equation in x

Obtain a correct horizontal equation free of surds

Solve for x and obtain answer $x=\sqrt{3}a$ **[6]**

28. M/J 13/P31/Q5

(i) Use correct quotient rule or equivalent

Obtain $\frac{(1+e^{2x})2x-(1+x^2)2e^{2x}}{(1+e^{2x})^2}$ or equivalent

Substitute $x=0$ and obtain $-\frac{1}{2}$ or equivalent [3]

(ii) Differentiate y^3 and obtain $3y^2\frac{dy}{dx}$

Differentiate $5xy$ and obtain $5y+5x\frac{dy}{dx}$

Obtain $6x^2+5y+5x\frac{dy}{dx}+3y^2\frac{dy}{dx}=0$

Substitute $x=0, y=2$ to obtain $-\frac{5}{6}$

or equivalent following correct work **[4]**

29. O/N 12/P32/Q7 , O/N 12/P31/Q7

(i) *EITHER*: State or imply $\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}$ as derivative of $\ln xy$, or equivalent [4]

State or imply $3y^2\frac{dy}{dx}$ as derivative of y^3, or equivalent

Equate derivative of LHS to zero and solve for $\frac{dy}{dx}$

Obtain the given answer

OR Obtain $xy = \exp(1+y^3)$ and state or imply $y + x\frac{dy}{dx}$ as derivative of xy

State or imply $3y^2\frac{dy}{dx}\exp(1+y^3)$ as derivative of $(1+y^3)$

Equate derivatives and solve for $\frac{dy}{dx}$

Obtain the given answer

[The M1 is dependent on at least one of the B marks being earned]

(ii) Equate denominator to zero and solve for y

Obtain $y = 0.693$ only [4]

Substitute found value in the equation and solve for x

Obtain $x = 5.47$ only

30. O/N 12/P33/Q3

(i) Either

Use correct quotient rule or equivalent to obtain

$\frac{dx}{dt}=\frac{4(2t+3)-8t}{(2t+3)^2}$ or equivalent

Obtain $\frac{dy}{dt}=\frac{4}{2t+3}$ or equivalent

Use $\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}$ or equivalent

Obtain $\frac{1}{3}(2t+3)$ or similarly simplified equivalent

Or

Express t in terms of x or y e.g. $t=\frac{3x}{4-2x}$

Obtain Cartesian equation e.g. $y=2\ln\left(\frac{6}{2-x}\right)$

Differentiate and obtain $\frac{dy}{dx}=\frac{2}{2-x}$

Obtain $\frac{1}{3}(2t+3)$ or similarly simplified equivalent [4]

(ii) Obtain $2t=3$ or $t=\frac{3}{2}$ [2]

Substitute in expression for $\frac{dy}{dx}$ and obtain 2

31. M/J 12/P31/Q6

(i) Obtain $2y\frac{dy}{dx}$ as derivative of y^2 [4]

Obtain $-4y-4x\frac{dy}{dx}$ as derivative of $-4xy$

Substitute $x = 2$ and $y = -3$ and find value of $\frac{dy}{dx}$

(dependent on at least one B1 being earned and $\frac{d(45)}{dx}=0$)

Obtain $\frac{12}{7}$ or equivalent

(ii) Substitute $\frac{dy}{dx}=1$ in an expression involving $\frac{dy}{dx}$, x and y and obtain $ay = bx$ [3]

Obtain $y = x$ or equivalent

Uses $y = x$ in original equation and demonstrate contradiction

32. M/J 12/P33/Q3

Obtain $\frac{dx}{d\theta} = 2\cos 2\theta - 1$ or $\frac{dy}{d\theta} = -2\sin 2\theta + 2\cos\theta$, or equivalent **[5]**

Use $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

Obtain $\frac{dy}{dx} = \frac{-2\sin 2\theta + 2\cos\theta}{2\cos 2\theta - 1}$, or equivalent

At any stage use correct double angle formulae throughout

Obtain the given answer following full and correct working

33. M/J 12/P33/Q4

(i) Use correct quotient or product rule **[4]**

Obtain correct derivative in any form, e.g. $\frac{2e^{2x}}{x^3} - \frac{3e^{2x}}{x^4}$

Equate derivative to zero and solve a 2-term equation for non-zero x

Obtain $x = \frac{3}{2}$ correctly

(ii) Carry out a method for determining the nature of a stationary point, e.g. test derivative either side **[2]**

Show point is a minimum with no errors seen

34. O/N 11/P32/Q2

EITHER :

Use chain rule

obtain $\frac{dx}{dt} = 6\sin t\cos t$, or equivalent

obtain $\frac{dy}{dt} = -6\cos^2 t\sin t$, or equivalent

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Obtain final answer $\frac{dy}{dx} = -\cos t$

OR :

Express y in terms of x and use chain rule

Obtain $\frac{dy}{dx} = k(2 - \frac{x}{3})^{\frac{1}{2}}$, or equivalent

Obtain $\frac{dy}{dx} = -(2 - \frac{x}{3})^{\frac{1}{2}}$, or equivalent

Express derivative in terms of t

Obtain final answer $\frac{dy}{dx} = -\cos t$ **[5]**

35. O/N 11/P33/Q2

Use correct quotient or product rule or equivalent **[4]**

Obtain $\frac{(1+e^{2x}).2e^{2x} - e^{2x}.2e^{2x}}{(1+e^{2x})^2}$ or equivalent

Substitute $x = \ln 3$ into attempt at first derivative and show use of relevant logarithm property at least once in a correct context

Confirm given answer $\frac{9}{50}$ legitimately

36. O/N 11/P33/Q8

(i)

Differentiate y to obtain $3\sin^2 t\cos t - 3\cos^2 t\sin t$ o.e. **[3]**

Use $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dt}{dx}$

Obtain given result $-3\sin t\cos t$

(ii) Identify parameter at origin as $t = \frac{3}{4}\pi$ **[2]**

Use $t = \frac{3}{4}\pi$ to obtain $\frac{3}{2}$

(iii) Rewrite equation as equation in one trig variable **[4]**

e.g. $\sin 2t = -\frac{2}{3}$, $9\sin^4 x - 9\sin^2 x + 1 = 0$, $\tan^2 x + 3\tan x + 1 = 0$

Find at least one value of t from equation of form $\sin 2t = k$ o.e.

Obtain 1.9

Obtain 2.8 and no others

37. M/J 11/P32/Q5

(i) *EITHER*: State $\frac{dx}{dt} = \sec^2 t / \tan t$, or equivalent [4]

State $\frac{dy}{dt} = 2\sin t \cos t$, or equivalent

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Obtain correct answer in any form, e.g. $2\sin^2 t \cos^2 t$

OR: Obtain $y = e^{2x} / (1 + e^{2x})$, or equivalent

Use correct quotient or product rule

Obtain correct derivative in any form, e.g. $2e^{2x} / (1 + e^{2x})^2$

Obtain correct derivative in terms of t in any form, e.g. $(2\tan^2 t) / (1 + \tan^2 t)^2$

(ii) State or imply $t = \frac{1}{4}\pi$ when $x = 0$ [3]

Form the equation of the tangent at $x = 0$

Obtain correct answer in any horizontal form, e.g. $y = \frac{1}{2}x + \frac{1}{2}$

[SR: If the *OR* method is used in part **(i)**, give B1 for stating or implying $y = \frac{1}{2}$ or $\frac{dy}{dx} = \frac{1}{2}$ when $x = 0$.]

38. M/J 11/P32/Q10

(i) Attempt integration by parts and reach $\pm x^2e^{-x} \pm \int 2xe^{-x}dx$ [5]

Obtain $-x^2e^{-x} + \int 2xe^{-x}dx$, or equivalent

Integrate and obtain $-x^2e^{-x} - 2xe^{-x} - 2e^{-x}$, or equivalent

Use limits $x = 0$ and $x = 3$, having integrated by parts twice

Obtain the given answer correctly

(ii) Use correct product or quotient rule [4]

Obtain correct derivative in any form

Equate derivative to zero and solve for non-zero x

Obtain $x = 2$ with no errors send

(iii) Carry out a complete method for finding the x-coordinate of P [2]

Obtain answer $x = 1$

39. M/J 11/P31/Q2

(i) Obtain $\frac{k\cos 2x}{1+\sin 2x}$ for any non-zero constant k [2]

Obtain $\frac{2\cos 2x}{1+\sin 2x}$

(ii) Use correct quotient or product rule [2]

Obtain $\frac{x\sec^2 x - \tan x}{x^2}$ or equivalent

40. M/J 11/P33/Q2

Use correct quotient or product rule [4]

Obtain correct derivative in any form, e.g. $-\frac{3\ln x}{x^4} + \frac{1}{x^4}$

Equate derivative to zero and solve for x an equation of the form $\ln x = a$, where $a > 0$

Obtain answer $\exp(\frac{1}{3})$, or 1.40, from correct work

41. O/N 10/P32/Q9 , O/N 10/P31/Q9

(i) Use correct product rule [5]

Obtain correct derivative in any form

Equate derivative to zero and find non-zero x

Obtain $x = \exp(-\frac{1}{3})$, or equivalent

Obtain $y = -1/(3e)$, or any ln-free equivalent

(ii) Integrate and reach $kx^4 \ln x + l\int x^4 . \frac{1}{x}\,dx$ [5]

Obtain $\frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx$

Obtain integral $\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4$, or equivalent

Use limits $x = 1$ and $x = 2$ correctly, having integrated twice

Obtain answer $4\ln 2 - \frac{15}{16}$, or exact equivalent

42. O/N 10/P33/Q2

Use of correct quotient or product rule to differentiate x or t [5]

Obtain correct $\frac{3}{(2t+3)^2}$ or unsimplified equivalent

Obtain $-2e^{-2t}$ for derivative of y

Use $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$ or equivalent

Obtain –6

Alternative:

Eliminate parameter and attempt differentiation $\left(y = e^{\frac{-6x}{1-2x}} \right)$

Use correct quotient or product rule

Use chain rule

Obtain $\frac{dy}{dx} = \frac{-6}{(1-2x)^2} e^{\frac{-6x}{1-2x}}$

Obtain –6

43. M/J 10/P32/Q6

(i) *EITHER*: State or imply $\frac{1}{y}\frac{dy}{dx}$ as derivative of $\ln y$ [4]

State correct derivative of LHS, e.g. $\ln y + \frac{x}{y}\frac{dy}{dx}$

Differentiate RHS and obtain an expression for $\frac{dy}{dx}$

Obtain given answer

***OR* 1**: State $\ln y = \frac{2x+1}{x}$, or equivalent, and differentiate both sides

State correct derivative of LHS, e.g. $\frac{1}{y}\frac{dy}{dx}$

State correct derivative of RHS, e.g. $-1/x^2$

Rearrange and obtain given answer

***OR* 2**: State $y = \exp(2 + 1/x)$, or equivalent, and attempt differentiation by chain rule

State correct derivative of RHS, e.g. $-\exp(2 + 1/x)/x^2$

Obtain given answer

[The B marks are for the exponential term and its multiplier.]

(ii) State or imply $x = -\frac{1}{2}$ when $y = 1$ [4]

Substitute and obtain gradient of –4

Correctly form equation of tangent

Obtain final answer $y + 4x + 1 = 0$, or equivalent

44. M/J 10/P31/Q9

(i) Use quotient or product rule to differentiate $(1-x)/(1+x)$ [5]
Obtain correct derivative in any form
Use chain rule to find $\frac{dy}{dx}$
Obtain a correct expression in any form
Obtain the gradient of the normal in the given form correctly

(ii) Use product rule
Obtain correct derivative in any form
Equate derivative to zero and solve for x
Obtain $x = \frac{1}{2}$ [4]

45. O/N 09/P32/Q3

(i) State $2xy + x^2 \frac{dy}{dx}$ as derivative of x^2y
State $3y^2 \frac{dy}{dx}$ as derivative of y^3
Equate derivative of LHS to zero and solve for $\frac{dy}{dx}$
Obtain answer $\frac{3x^2 - 2xy}{x^2 + 3y^2}$, or equivalent [4]

(ii) Find gradient of tangent at (2, 1) and form equation of tangent
Obtain answer $8x - 7y - 9 = 0$, or equivalent [2]

46. O/N 09/P31/Q4

Use product or quotient rule
Obtain derivative in any correct form
Equate derivative to zero and obtain an equation of the form $a \sin 2x = b$, or a quadratic in $\tan x$, $\sin^2 x$, or $\cos^2 x$
Carry out correct method for finding one angle
Obtain answer, e.g. 0.365
Obtain second answer 1.206 and no others in the range (allow 1.21) [6]
[Ignore answers outside the given range.]
[Treat answers in degrees, 20.9° and 69.1°, as a misread.]

47. O/N 09/P31/Q9

(i) State coordinates (1, 0) [1]

(ii) Use correct quotient or product rule
Obtain derivative in any correct form
Equate derivative to zero and solve for x
Obtain $x = e^2$ correctly [4]

(iii) Attempt integration by parts reaching $a\sqrt{x}\ln x \pm a\int \sqrt{x}\frac{1}{x}dx$
Obtain $2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}dx$
Integrate and obtain $2\sqrt{x}\ln x - 4\sqrt{x}$
Use limits $x = 1$ and $x = 4$ correctly, having integrated twice
Justify the given answer [5]

48. O/N 08/P03/Q3

Use correct quotient or product rule [5]
Obtain correctly the derivative in any form, e.g. $\frac{e^x \cos x + e^x \sin x}{\cos^2 x}$
Equate derivative to zero and reach $\tan x = k$
Solve for x
Obtain $x = -\frac{1}{4}\pi$ (or -0.785) only (accept x in $[-0.79, -0.78]$ but not in degrees)

[The last three marks are independent. Fallacious log work forfeits the M1*. For the M1(dep*) the solution can lie outside the given range and be in degrees, but the mark is not available if $k = 0$. The final A1 is only given for an entirely correct answer to the whole question.]

49. O/N 08/P03/Q4

State or imply $\frac{dx}{d\theta} = a(2 - 2\cos 2\theta)$ or $\frac{dy}{d\theta} = 2a\sin 2\theta$ [5]

Use $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

Obtain $\frac{dy}{dx} = \frac{\sin 2\theta}{(1 - \cos 2\theta)}$, or equivalent

Make use of correct sin $2A$ and cos $2A$ formulae

Obtain the given result following sufficient working

[SR: An attempt which assumes a is the parameter and θ a constant can only earn the two M marks. One that assumes θ is the parameter and a is a function of θ can earn B1M1A0M1A0.]

[SR: For an attempt that gives a a value, e.g. 1, or ignores a, give B0 but allow the remaining marks.]

50. M/J 08/P03/Q6

EITHER State $x^2\frac{dy}{dx} + 2xy$, or equivalent, as derivative of x^2y [8]

State $y^2 + 2xy\frac{dy}{dx}$, or equivalent, as derivative of xy^2

OR State $xy(1 + \frac{dy}{dx})$, or equivalent, as a term in an attempt to apply the product rule

State $(y + x\frac{dy}{dx})(x + y)$, or equivalent, in an attempt to apply the product rule

Equate attempted derivative of LHS to zero and set $\frac{dy}{dx}$ equal to zero

Obtain a horizontal equation, e.g. $y^2 = -2xy$, or $y = -2x$, or equivalent

Explicitly reject $y = 0$ as a possibility

Obtain an equation in x (or in y)

Obtain $x = a$

Obtain $y = -2a$ only

[The first M1 is dependent on at least one B mark having been earned.]

[SR: for an attempt using $(x + y) = 2a^3 / xy$, the B marks are given for the correct derivatives of the two sides of the equation, and the M1 for setting $\frac{dy}{dx}$ equal to zero.]

[SR: for an attempt which begins by expressing y in terms of x, give M1A1 for a reasonable attempt at differentiation, M1A1√ for setting $\frac{dy}{dx}$ equal to zero and obtaining an equation free of surds, A1 for solving and obtaining $x = a$; then M1 for obtaining an equation for y, A1 for $y = -2a$ and A1 for finding and rejecting $y = a$ as a possibility.]

51. O/N 07/P03/Q4

(i) Use correct product or quotient rule [4]

Obtain derivative in any correct form

Equate derivative to zero and solve for x

Obtain answer $x = \frac{1}{4}\pi$ or 0.785 with no errors seen

(ii) Use an appropriate method for determining the nature of a stationary point [2]

Show the point is a maximum point with no errors seen

[SR: for the answer 45° deduct final A1 in part (i), and deduct A1 in part (ii) if this value in degrees is used in the exponential.]

52. M/J 07/P03/Q3

Use product rule

Obtain derivative in any correct form 4

Form equation of tangent at $x = \frac{1}{4}\pi$ correctly

Simplify answer to $y = x$, or $y - x = 0$

[SR : The misread $y - x\sin x$ can only earn M1 M1.]

53. O/N 06/P03/Q3

(i) State derivative is $6e^{x} - 3e^{3x}$ 4

EITHER : Equate derivative to zero and simplify to an equation of the form $e^{2x} = a$

Carry out method for calculating x, where $a > 0$

Obtain answer $x = \frac{1}{2}\ln 2$, or equivalent (0.347, or 0.346, or 0.35)

OR : Equate terms of the derivative and obtain a linear equation in x by taking logs correctly

Solve the linear equation for x

Obtain answer $x = \frac{1}{2}\ln 2$, or equivalent (0.347, or 0.346, or 0.35)

(ii) Carry out a method for determining the nature of a stationary point 2

Show that the point is a maximum with no errors seen.

54. O/N 06/P03/Q6

(i) State $2(3y^2)\frac{dy}{dx}$ as derivative of $2y^3$, or equivalent 4

State $3x\frac{dy}{dx} + 3y$ as derivative of $3xy$, or equivalent

Solve for $\frac{dy}{dx}$

Obtain given answer correctly

[The M1 is dependent on at least one of the B marks being obtained.]

(ii) State or imply that the coordinates satisfy $y - x^2 = 0$ 5

Obtain an equation in x (or in y)

Solve and obtain $x - 1$ only (or $y - 1$ only)

Substitute x=(or y=) value in $y - x^2 = 0$ or in the equation of the curve

Obtain $y = 1$ only (or $x = 1$ only)

[SR : If B1 is earned and (1 , 1) stated to be the only solution with no other evidence, award B2. If the point is also shown to lie on the curve award a further B2.]

55. M/J 06/P03/Q3

State that $\frac{dx}{d\theta} = 2 + 2\cos 2\theta$ or $\frac{dy}{d\theta} = 2\sin 2\theta$ [5]

Use $\frac{dy}{dx} = \frac{dy}{d\theta} + \frac{dx}{d\theta}$

Obtain answer in any correct form, e.g $\frac{2\sin 2\theta}{2+2\cos 2\theta}$

Make relevant use of sin 2A and cos 2A formulac

Obtain given answer correctly.

56. O/N 05/P03/Q3

State correct derivative $1 - 2\sin 2x$

Equate derivative to zero and solve for x

Obtain answer $x = \frac{1}{12}\pi$

Carry out an appropriate method for determining the nature of a stationary point

Show that $x = \frac{1}{12}\pi$ is a maximum with no errors seen

Obtain second answer $x = \frac{5}{12}\pi$ in range

Show this is a minimum point

57. M/J 04/P03/Q3

EITHER: State $6y\frac{dy}{dx}$ as the derivative of $3y^2$ 4

State $\pm 4x\frac{dy}{dx} \pm 4y$ as the derivative of $-4xy$

Equate attempted derivative of LHS to zero and solve for $\frac{dy}{dx}$

Obtain answer 2

[The M1 is conditional on at least one of the B marks being obtained. Allow any combination of signs for the second B1.]

OR: Obtain a correct expression for y in terms of x

Differentiate using chain rule

Obtain derivative in any correct form

Substitute $x = 2$ and obtain answer 2 only

[The M1 is conditional on a reasonable attempt at solving the quadratic in y being made.]

58. O/N 03/P03/Q4

(i) *EITHER* Obtain terms $\frac{1}{2\sqrt{x}}$ and $\frac{1}{2\sqrt{y}}\frac{dy}{dx}$, or equivalent [3]

Obtain answer in any correct form, e.g. $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

OR: Using chain or product rule, differentiate $(\sqrt{a} - \sqrt{x})^2$

Obtain derivative in any correct form

Express $\frac{dy}{dx}$ in terms of x and y only in any correct form

OR Expand $(\sqrt{a} - \sqrt{x})^2$, differentiate and obtain term $-2.\frac{\sqrt{a}}{2\sqrt{x}}$, or equivalent

Obtain term 1 by differentiating an expansion of the form $a + x \pm 2\sqrt{a}\sqrt{x}$

Express $\frac{dy}{dx}$ in terms of x and y only in any correct form

(ii) State or imply coordinates of P are $(\frac{1}{4}a, \frac{1}{4}a)$ [3]

Form equation of the tangent at P

Obtain 3 term answer $x + y = \frac{1}{2}a$ correctly, or equivalent

59. O/N 02/P03/Q4

(i) Obtain derivative $e^x - 8e^{-2x}$ in any correct form 4

Equate derivative to zero and simplify to an equation of the form $e^{kx} = a$, where $a \neq 0$.

Carry out method for calculating x with $a > 0$

Obtain answer $x = \ln 2$, or an exact equivalent (also accept 0.693 or 0.69)

[Accept statements of the form ' $u^k = a$, where $u = e^x$ ' for the first M1.]

(ii) Carry out a method for determining the nature of the stationary point 2

Show that the point is a minimum correctly, with no incorrect work seen.

60. M/J 02/P03/Q5

Obtain derivative $\pm 2\sin x + k\cos 2x$ or $\pm 2\sin x + k(\cos^2 x \pm \sin^2 x)$ 7

Equate derivative to zero and use trig formula to obtain an equation involving only one trig function

Obtain a correct equation of this type e.g $2\sin^2 x + \sin x - 1 = 0$ or $\cos 2x = \cos(\frac{1}{2}\pi - x)$

Obtain value $x = \frac{1}{6}\pi$ (allow 0.524 radians or 30°)

Show by any method that the corresponding point is a maximum point

Obtain second value $x = \frac{5}{6}\pi$ (allow 2.62 radians or 150°). and no others in range

Determine that is corresponds to a minimum point.

UNIT 5

Integration

A-Level
Mathematics Paper 3
Topical Workbook

Topics

READ & WRITE PUBLICATIONS
Sale Point: Shop # 25-28 Lower Ground Floor,
Haadia Haleema Centre, Ghazni Street,
Urdu Bazar, Lahore.
Tel: +92 42-35714038
Cell: +92 336-5314141
Web: www.readnwrite.org
E-mail: readandwriteoffice@gmail.com

Unit-5: Integration

5.1: Integration

1. M/J 18/P32/Q4(ii)

(i) Hence, showing all necessary working, find $\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{2\sin x - \sin 2x}{1-\cos 2x}\,dx$, giving your answer in the form $\ln k$. [4]

2. M/J 18/P31/Q5

Let $I = \int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\,dx$.

(i) Using the substitution $x = \cos^2\theta$, show that $I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2\cos^2\theta\,d\theta$. [4]

(ii) Hence find the exact value of I. [4]

3. M/J 18/P33/Q3

Showing all necessary working, find the value of $\int_0^{\frac{1}{6}\pi} x\cos 3x\,dx$, giving your answer in terms of π. [5]

4. M/J 18/P33/Q7(ii)

(i) Hence, showing all necessary working, show that $\int_0^{\frac{1}{4}\pi} \frac{15}{(\cos\theta + 2\sin\theta)^2}\,d\theta = 5$. [5]

5. O/N 17/P32/Q9

It is given that $\int_1^a x^{\frac{1}{2}}\ln x\,dx = 2$, where $a > 1$.

(i) Show that $a^{\frac{3}{2}} = \frac{7 + 2a^{\frac{3}{2}}}{3\ln a}$. [5]

(ii) Show by calculation that a lies between 2 and 4. [2]

(iii) Use the iterative formula

$$a_{n+1} = \left(\frac{7 + 2a_n^{\frac{3}{2}}}{3\ln a_n}\right)^{\frac{2}{3}}$$

to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

6. O/N 17/P31/Q8, O/N 17/P31/Q8

Let $f(x) = \frac{4x^2 + 9x - 8}{(x+2)(2x-1)}$.

(i) Express $f(x)$ in the form $A + \frac{B}{x+2} + \frac{C}{2x-1}$. [4]

(ii) Hence show that $\int_1^4 f(x)\,dx = 6 + \frac{1}{2}\ln\left(\frac{16}{7}\right)$. [5]

7. M/J 17/P31/Q3(ii)

It is given that $x = \ln(1 - y) - \ln y$, where $0 < y < 1$.

(i) Hence show that $\int_0^1 y \, dx = \ln\left(\frac{2e}{e+1}\right)$. **[4]**

8. M/J 17/P31/Q9

(i) Express $\frac{1}{x(2x+3)}$ in partial fractions. **[2]**

(ii) The variables x and y satisfy the differential equation

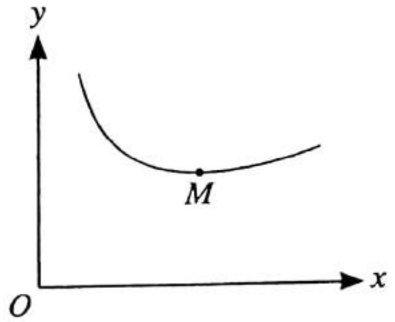
$$x(2x+3)\frac{dy}{dx} = y,$$

and it is given that $y = 1$ when $x = 1$. Solve the differential equation and calculate the value of y when $x = 9$, giving your answer correct to 3 significant figures. **[7]**

9. M/J 17/P33/Q4

Find the exact value of $\int_0^{\frac{1}{2}\pi} \theta \sin \frac{1}{2}\theta \, d\theta$. **[4]**

10. M/J 17/P33/Q7(i,ii)



The diagram shows a sketch of the curve $y = \frac{e^{\frac{1}{2}x}}{x}$ for $x > 0$, and its minimum point M.

(i) Find the x-coordinate of M. **[4]**

(ii) Use the trapezium rule with two intervals to estimate the value of

$$\int_1^3 \frac{e^{\frac{1}{2}x}}{x} \, dx,$$

giving your answer correct to 2 decimal places. **[3]**

11. M/J 17/P33/Q9(ii)

Let $f(x) = \frac{3x^2 - 4}{x^2(3x+2)}$.

(i) Hence show that $\int_1^2 f(x) \, dx = \ln\left(\frac{25}{8}\right) - 1$. **[5]**

12. O/N 16/P32/Q5, O/N 16/P31/Q5

(i) Prove the identity $\tan 2\theta - \tan\theta \equiv \tan\theta \sec 2\theta$. **[4]**

(ii) Hence show that $\int_0^{\frac{1}{6}\pi} \tan\theta \sec 2\theta \, d\theta = \frac{1}{2}\ln\frac{3}{2}$. **[4]**

13. O/N 16/P33/Q6

Let $I = \displaystyle\int_1^4 \frac{(\sqrt{x}) - 1}{2(x + \sqrt{x})}\,dx$.

(i) Using the substitution $u = \sqrt{x}$, show that $I = \displaystyle\int_1^2 \frac{u-1}{u+1}\,du$. [3]

(ii) Hence show that $I = 1 + \ln\frac{4}{9}$. [6]

14. M/J 16/P32/Q3

Find the exact value of $\displaystyle\int_0^{\frac{1}{2}\pi} x^2 \sin 2x\,dx$. [5]

15. M/J 16/P32/Q7

Let $f(x) = \dfrac{4x^2 + 7x + 4}{(2x+1)(x+2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that $\displaystyle\int_0^4 f(x)\,dx = 8 - \ln 3$. [5]

16. M/J 16/P31/Q2

Find the exact value of $\displaystyle\int_0^{\frac{1}{2}} xe^{-2x}\,dx$. [5]

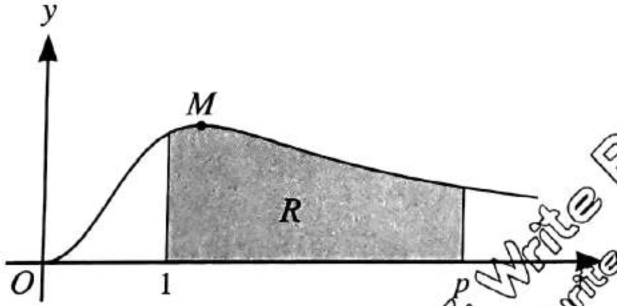
17. M/J 16/P33/Q7

Let $I = \displaystyle\int_0^1 \frac{x^5}{(1+x^2)^3}\,dx$.

(i) Using the substitution $u = 1 + x^2$, show that $I = \displaystyle\int_1^2 \frac{(u-1)^2}{2u^3}\,du$. [3]

(ii) Hence find the exact value of I. [5]

18. O/N 15/P32/Q10, O/N 15/P31/Q10



The diagram shows the curve $y = \dfrac{x^2}{1+x^3}$ for $x \geqslant 0$, and its maximum point M. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 1$ and $x = p$.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Calculate the value of p for which the area of R is equal to 1. Give your answer correct to 3 significant figures. [6]

19. O/N 15/P33/Q5

Use the substitution $u = 4 - 3\cos x$ to find the exact value of $\int_0^{\frac{1}{2}\pi} \frac{9\sin 2x}{\sqrt{(4-3\cos x)}}\,dx$. **[8]**

20. O/N 15/P33/Q7

(i) Show that $(x+1)$ is a factor of $4x^3 - x^2 - 11x - 6$. **[2]**

(ii) Find $\int \frac{4x^2+9x-1}{4x^3-x^2-11x-6}\,dx$. **[8]**

21. M/J 15/P32/Q6

Let $I = \int_0^1 \frac{\sqrt{x}}{2-\sqrt{x}}\,dx$.

(i) Using the substitution $u = 2 - \sqrt{x}$, show that $I = \int_1^2 \frac{2(2-u)^2}{u}\,du$. **[4]**

(ii) Hence show that $I = 8\ln 2 - 5$. **[4]**

22. M/J 15/P31/Q5

(a) Find $\int (4 + \tan^2 2x)\,dx$. **[3]**

(b) Find the exact value of $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{\sin(x+\frac{1}{6}\pi)}{\sin x}\,dx$. **[5]**

23. M/J 15/P33/Q6

It is given that $\int_0^a x\cos x\,dx = 0.5$, where $0 < a < \frac{1}{2}\pi$.

(i) Show that a satisfies the equation $\sin a = \frac{1.5 - \cos a}{a}$. **[4]**

(ii) Verify by calculation that a is greater than 1. **[2]**

(iii) Use the iterative formula

$$a_{n+1} = \sin^{-1}\left(\frac{1.5 - \cos a_n}{a_n}\right)$$

to determine the value of a correct to 4 decimal places, giving the result of each iteration to 6 decimal places. **[3]**

24. M/J 15/P33/Q10

Let $f(x) = \frac{11x+7}{(2x-1)(x+2)^2}$.

(i) Express $f(x)$ in partial fractions. **[5]**

(ii) Show that $\int_1^2 f(x)\,dx = \frac{1}{4} + \ln(\frac{9}{4})$. **[5]**

25. O/N 14/P32/Q2, O/N 14/P31/Q2

(i) Use the trapezium rule with 3 intervals to estimate the value of

$$\int_{\frac{1}{6}\pi}^{\frac{2}{3}\pi} \operatorname{cosec} x\,dx,$$

giving your answer correct to 2 decimal places. **[3]**

(ii) Using a sketch of the graph of $y = \operatorname{cosec} x$, explain whether the trapezium rule gives an overestimate or an underestimate of the true value of the integral in part **(i)**. [2]

26. O/N 14/P32/Q6, O/N 14/P31/Q6

It is given that $\int_1^a \ln(2x)\,dx = 1$, where $a > 1$.

(i) Show that $a = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a}\right)$, where $\exp(x)$ denotes e^x. [6]

(ii) Use the iterative formula

$$a_{n+1} = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a_n}\right)$$

to determine the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

27. O/N 14/P33/Q6

It is given that $I = \int_0^{0.3} (1 + 3x^2)^{-2}\,dx$.

(i) Use the trapezium rule with 3 intervals to find an approximation to I, giving the answer correct to 3 decimal places. [3]

(ii) For small values of x, $(1 + 3x^2)^{-2} \approx 1 + ax^2 + bx^4$. Find the values of the constants a and b. Hence, by evaluating $\int_0^{0.3} (1 + ax^2 + bx^4)\,dx$, find a second approximation to I, giving the answer correct to 3 decimal places. [5]

28. O/N 14/P33/Q10

By first using the substitution $u = e^x$, show that

$$\int_0^{\ln 4} \frac{e^{2x}}{e^{2x} + 3e^x + 2}\,dx = \ln\left(\tfrac{8}{5}\right).$$ [10]

29. M/J 14/P31/Q2

Use the substitution $u = 1 + 3\tan x$ to find the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sqrt{(1 + 3\tan x)}}{\cos^2 x}\,dx.$$ [5]

30. M/J 14/P33/Q8

Let $f(x) = \dfrac{6 + 6x}{(2 - x)(2 + x^2)}$.

(i) Express $f(x)$ in the form $\dfrac{A}{2 - x} + \dfrac{Bx + C}{2 + x^2}$. [4]

(ii) Show that $\int_{-1}^{1} f(x)\,dx = 3\ln 3$. [5]

31. O/N 13/P32/Q3

Find the exact value of $\int_1^4 \dfrac{\ln x}{\sqrt{x}}\,dx$. [5]

32. O/N 13/P32/Q5

(i) Prove that $\cot\theta + \tan\theta \equiv 2\operatorname{cosec} 2\theta$. **[3]**

(ii) Hence show that $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \operatorname{cosec} 2\theta \, d\theta = \frac{1}{2}\ln 3$. **[4]**

33. O/N 13/P33/Q2

Use the substitution $u = 3x + 1$ to find $\int \frac{3x}{3x+1}\, dx$. **[4]**

34. O/N 13/P33/Q5

It is given that $\int_0^p 4xe^{-\frac{1}{2}x}\, dx = 9$, where p is a positive constant.

(i) Show that $p = 2\ln\left(\frac{8p+16}{7}\right)$. **[5]**

(ii) Use an iterative process based on the equation in part (i) to find the value of p correct to 3 significant figures. Use a starting value of 3.5 and give the result of each iteration to 5 significant figures. **[3]**

35. M/J 13/P32/Q6

(i) By differentiating $\frac{1}{\cos x}$, show that the derivative of $\sec x$ is $\sec x \tan x$. Hence show that if $y = \ln(\sec x + \tan x)$ then $\frac{dy}{dx} = \sec x$. **[4]**

(ii) Using the substitution $x = (\sqrt{3})\tan\theta$, find the exact value of

$$\int_1^3 \frac{1}{\sqrt{(3+x^2)}}\, dx,$$

expressing your answer as a single logarithm. **[4]**

36. M/J 13/P31/Q8

(a) Show that $\int_2^4 4x\ln x\, dx = 56\ln 2 - 12$. **[5]**

(b) Use the substitution $u = \sin 4x$ to find the exact value of $\int_0^{\frac{1}{24}\pi} \cos^3 4x\, dx$. **[5]**

37. M/J 13/P31/Q9

(i) Express $4\cos\theta + 3\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the value of α correct to 4 decimal places. **[3]**

(ii) Hence

(a) solve the equation $4\cos\theta + 3\sin\theta = 2$ for $0 < \theta < 2\pi$, **[4]**

(b) find $\int \frac{50}{(4\cos\theta + 3\sin\theta)^2}\, d\theta$. **[3]**

38. M/J 13/P33/Q4

(i) Express $(\sqrt{3})\cos x + \sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. **[3]**

(ii) Hence show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \frac{1}{((\sqrt{3})\cos x + \sin x)^2}\,dx = \tfrac{1}{4}\sqrt{3}.$$ [4]

39. O/N 12/P32/Q5, O/N 12/P31/Q5

(i) By differentiating $\frac{1}{\cos x}$, show that if $y = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$. [2]

(ii) Show that $\frac{1}{\sec x - \tan x} \equiv \sec x + \tan x$. [1]

(iii) Deduce that $\frac{1}{(\sec x - \tan x)^2} \equiv 2\sec^2 x - 1 + 2\sec x \tan x$. [2]

(iv) Hence show that $\int_0^{\frac{1}{4}\pi} \frac{1}{(\sec x - \tan x)^2}\,dx = \frac{1}{4}(8\sqrt{2} - \pi)$. [3]

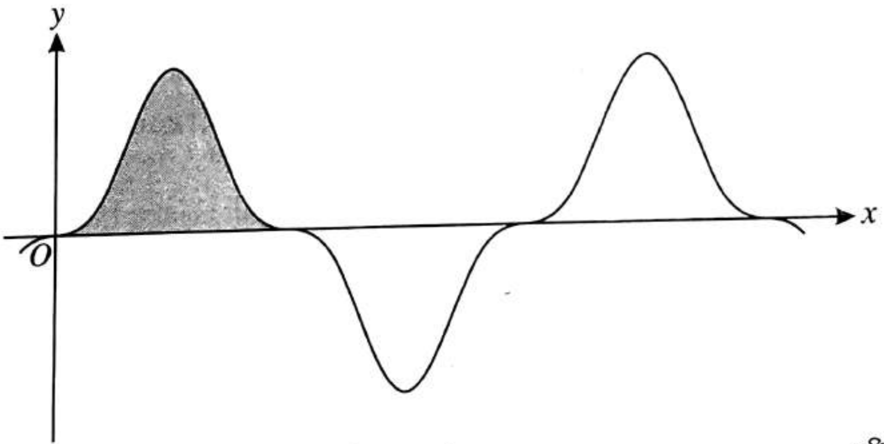
40. O/N 12/P33/Q5

The expression $f(x)$ is defined by $f(x) = 3xe^{-2x}$.

(i) Find the exact value of $f'(-\frac{1}{2})$. [3]

(ii) Find the exact value of $\int_{-\frac{1}{2}}^{0} f(x)\,dx$. [5]

41. O/N 12/P33/Q7



The diagram shows part of the curve $y = \sin^3 2x \cos^3 2x$. The shaded region shown is bounded by the curve and the x-axis and its exact area is denoted by A.

(i) Use the substitution $u = \sin 2x$ in a suitable integral to find the value of A. [6]

(ii) Given that $\int_0^{k\pi} |\sin^3 2x \cos^3 2x|\,dx = 40A$, find the value of the constant k. [2]

42. M/J 12/P32/Q8

Let $I = \int_2^5 \frac{5}{x + \sqrt{(6-x)}}\,dx$.

(i) Using the substitution $u = \sqrt{(6-x)}$, show that

$$I = \int_1^2 \frac{10u}{(3-u)(2+u)}\,du.$$ [4]

(ii) Hence show that $I = 2\ln(\frac{9}{2})$. [6]

43. M/J 12/P31/Q9

By first expressing $\dfrac{4x^2+5x+3}{2x^2+5x+2}$ in partial fractions, show that

$$\int_0^4 \frac{4x^2+5x+3}{2x^2+5x+2}\,dx = 8 - \ln 9.$$ [10]

44. M/J 12/P33/Q8

Let $f(x) = \dfrac{4x^2-7x-1}{(x+1)(2x-3)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that $\int_2^6 f(x)\,dx = 8 - \ln\left(\frac{49}{3}\right)$. [5]

45. O/N 11/P32/Q8, O/N 11/P31/Q8

Let $f(x) = \dfrac{12+8x-x^2}{(2-x)(4+x^2)}$.

(i) Express $f(x)$ in the form $\dfrac{A}{2-x} + \dfrac{Bx+C}{4+x^2}$. [4]

(ii) Show that $\int_0^1 f(x)\,dx = \ln\left(\frac{25}{2}\right)$. [5]

46. O/N 11/P33/Q10

(i) Use the substitution $u = \tan x$ to show that, for $n \neq -1$,

$$\int_0^{\frac{1}{4}\pi} (\tan^{n+2} x + \tan^n x)\,dx = \frac{1}{n+1}.$$ [4]

(ii) Hence find the exact value of

(a) $\int_0^{\frac{1}{4}\pi} (\sec^4 x - \sec^2 x)\,dx$, [3]

(b) $\int_0^{\frac{1}{4}\pi} (\tan^9 x + 5\tan^7 x + 5\tan^5 x + \tan^3 x)\,dx$. [3]

47. M/J 11/P31/Q7

The integral I is defined by $I = \int_0^2 4t^3 \ln(t^2+1)\,dt$.

(i) Use the substitution $x = t^2 + 1$ to show that $I = \int_1^5 (2x-2)\ln x\,dx$. [3]

(ii) Hence find the exact value of I. [5]

48. M/J 11/P31/Q9

(i) Prove the identity $\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3$. [4]

(ii) Hence

(a) solve the equation $\cos 4\theta + 4\cos 2\theta = 1$ for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, [3]

(b) find the exact value of $\int_0^{\frac{1}{4}\pi} \cos^4\theta\,d\theta$. [3]

49. M/J 11/P33/Q3

Show that $\int_0^1 (1-x)e^{-\frac{1}{2}x}\,dx = 4e^{-\frac{1}{2}} - 2.$ [5]

50. O/N 10/P32/Q5, O/N 10/P31/Q5

Let $I = \int_0^1 \frac{x^2}{\sqrt{(4-x^2)}}\,dx.$

(i) Using the substitution $x = 2\sin\theta$, show that

$$I = \int_0^{\frac{1}{6}\pi} 4\sin^2\theta\,d\theta.$$ [3]

(ii) Hence find the exact value of I. [4]

51. O/N 10/P33/Q4

It is given that $f(x) = 4\cos^2 3x$.

(i) Find the exact value of $f'(\frac{1}{9}\pi)$. [3]

(ii) Find $\int f(x)\,dx$. [3]

52. O/N 10/P33/Q5

Show that $\int_0^7 \frac{2x+7}{(2x+1)(x+2)}\,dx = \ln 50.$ [7]

53. M/J 10/P32/Q2

Show that $\int_0^{\pi} x^2 \sin x\,dx = \pi^2 - 4.$ [5]

54. M/J 10/P32/Q10

(i) Find the values of the constants A, B, C and D such that

$$\frac{2x^3-1}{x^2(2x-1)} \equiv A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{2x-1}.$$ [5]

(ii) Hence show that

$$\int_1^2 \frac{2x^3-1}{x^2(2x-1)}\,dx = \frac{3}{2} + \frac{1}{2}\ln\left(\frac{16}{27}\right).$$ [5]

55. M/J 10/P31/Q4

(i) Using the expansions of $\cos(3x-x)$ and $\cos(3x+x)$, prove that

$$\frac{1}{2}(\cos 2x - \cos 4x) \equiv \sin 3x \sin x.$$ [3]

(ii) Hence show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin 3x \sin x\,dx = \frac{1}{8}\sqrt{3}.$$ [3]

56. M/J 10/P31/Q8

(i) Express $\frac{2}{(x+1)(x+3)}$ in partial fractions. [2]

(ii) Using your answer to part **(i)**, show that

$$\left(\frac{2}{(x+1)(x+3)}\right)^2 \equiv \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+3} + \frac{1}{(x+3)^2}.$$ **[2]**

(iii) Hence show that $\displaystyle\int_0^1 \frac{4}{(x+1)^2(x+3)^2}\,dx = \frac{7}{12} - \ln\frac{3}{2}$. **[5]**

57. M/J 10/P33/Q7

(i) Prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. **[4]**

(ii) Using this result, find the exact value of

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \cos^3\theta\,d\theta.$$ **[4]**

58. O/N 09/P32/Q6

(i) Use the substitution $x = 2\tan\theta$ to show that

$$\int_0^2 \frac{8}{(4+x^2)^2}\,dx = \int_0^{\frac{1}{4}\pi} \cos^2\theta\,d\theta.$$ **[4]**

(ii) Hence find the exact value of

$$\int_0^2 \frac{8}{(4+x^2)^2}\,dx.$$ **[4]**

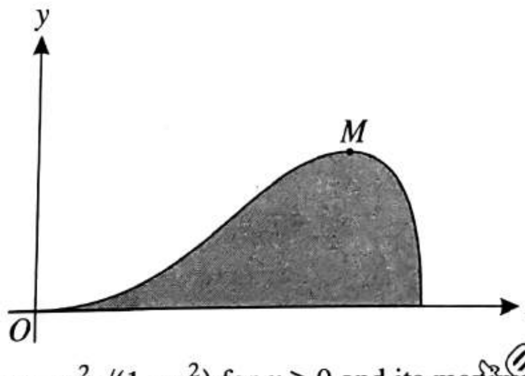
59. O/N 09/P31/Q5

(i) Prove the identity $\cos 4\theta - 4\cos 2\theta + 3 \equiv 8\sin^4\theta$. **[4]**

(ii) Using this result find, in simplified form, the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin^4\theta\,d\theta.$$ **[4]**

60. M/J 09/P3/Q10



The diagram shows the curve $y = x^2\sqrt{(1-x^2)}$ for $x \geqslant 0$ and its maximum point M.

(i) Find the exact value of the x-coordinate of M. **[4]**

(ii) Show, by means of the substitution $x = \sin\theta$, that the area A of the shaded region between the curve and the x-axis is given by

$$A = \frac{1}{4}\int_0^{\frac{1}{2}\pi} \sin^2 2\theta\,d\theta.$$ **[3]**

(iii) Hence obtain the exact value of A. **[4]**

61. O/N 08/P3/Q9

The constant a is such that $\int_0^a xe^{\frac{1}{2}x}\,dx = 6$.

(i) Show that a satisfies the equation

$$x = 2 + e^{-\frac{1}{2}x}.$$ [5]

(ii) By sketching a suitable pair of graphs, show that this equation has only one root. [2]

(iii) Verify by calculation that this root lies between 2 and 2.5. [2]

(iv) Use an iterative formula based on the equation in part (i) to calculate the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

62. M/J 08/P3/Q7

Let $f(x) \equiv \dfrac{x^2 + 3x + 3}{(x+1)(x+3)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_0^3 f(x)\,dx = 3 - \frac{1}{2}\ln 2$. [4]

63. O/N 07/P3/Q1

Find the exact value of the constant k for which $\int_1^k \dfrac{1}{2x-1}\,dx = 1$. [4]

64. O/N 07/P3/Q3

Use integration by parts to show that

$$\int_2^4 \ln x\,dx = 6\ln 2 - 2.$$ [4]

65. M/J 07/P3/Q5

(i) Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

(ii) Hence show that $\int_0^{\frac{1}{2}\pi} \dfrac{1}{(\cos\theta + (\sqrt{3})\sin\theta)^2}\,d\theta = \dfrac{1}{\sqrt{3}}$. [4]

66. M/J 07/P3/Q7

Let $I = \int_1^4 \dfrac{1}{x(4-\sqrt{x})}\,dx$.

(i) Use the substitution $u = \sqrt{x}$ to show that $I = \int_1^2 \dfrac{2}{u(4-u)}\,du$. [3]

(ii) Hence show that $I = \frac{1}{2}\ln 3$. [6]

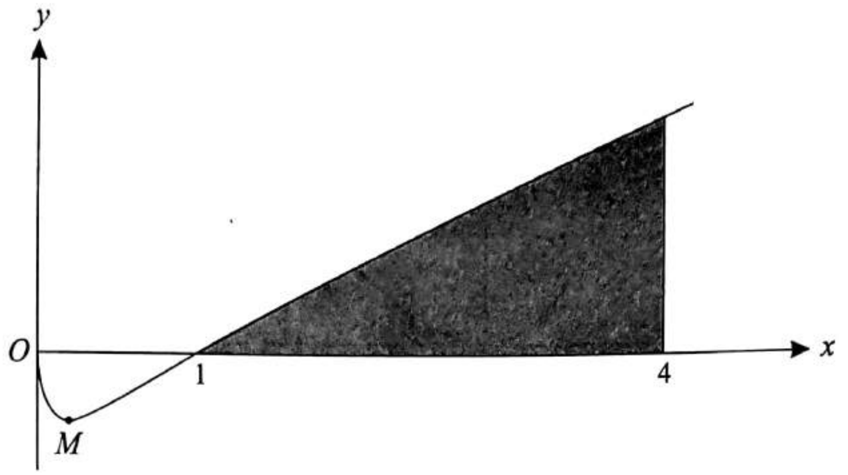
67. O/N 06/P3/Q8

Let $f(x) = \dfrac{7x+4}{(2x+1)(x+1)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_0^2 f(x)\,dx = 2 + \ln\frac{5}{3}$. [5]

68. M/J 06/P3/Q8



The diagram shows a sketch of the curve $y = x^{\frac{1}{2}} \ln x$ and its minimum point M. The curve cuts the x-axis at the point $(1, 0)$.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Use integration by parts to find the area of the shaded region enclosed by the curve, the x-axis and the line $x = 4$. Give your answer correct to 2 decimal places. [5]

69. O/N 05/P3/Q6

(i) Use the substitution $x = \sin^2 \theta$ to show that

$$\int \sqrt{\left(\frac{x}{1-x}\right)}\, dx = \int 2\sin^2 \theta \, d\theta.$$ [4]

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\, dx.$$ [4]

70. M/J 05/P3/Q4

(i) Use the substitution $x = \tan \theta$ to show that

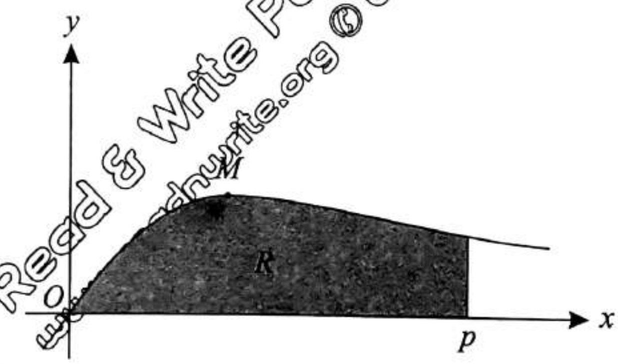
$$\int \frac{1-x^2}{(1+x^2)^2}\, dx = \int \cos 2\theta \, d\theta.$$ [4]

(ii) Hence find the value of

$$\int_0^1 \frac{1-x^2}{(1+x^2)^2}\, dx.$$ [3]

71. M/J 05/P3/Q9

The diagram shows part of the curve $y = \dfrac{x}{x^2+1}$ and its maximum point M. The shaded region R is bounded by the curve and by the lines $y = 0$ and $x = p$.

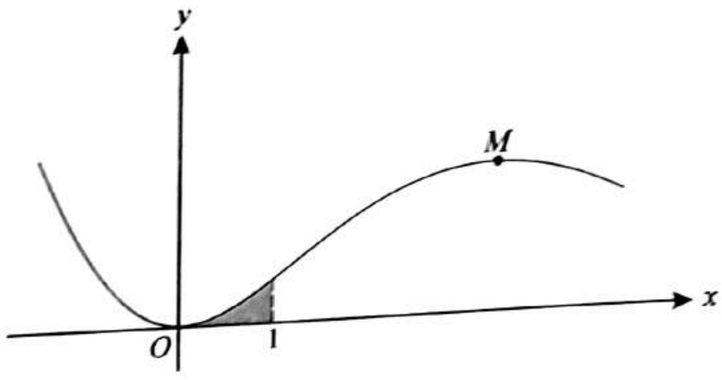


(i) Calculate the x-coordinate of M. [4]

(ii) Find the area of R in terms of p. [3]

(iii) Hence calculate the value of p for which the area of R is 1, giving your answer correct to 3 significant figures. [2]

72. O/N 04/P3/Q7



The diagram shows the curve $y = x^2 e^{-\frac{1}{2}x}$.

(i) Find the x-coordinate of M, the maximum point of the curve. [4]

(ii) Find the area of the shaded region enclosed by the curve, the x-axis and the line $x = 1$, giving your answer in terms of e. [5]

73. O/N 04/P3/Q8

An appropriate form for expressing $\dfrac{3x}{(x+1)(x-2)}$ in partial fractions is

$$\frac{A}{x+1} + \frac{B}{x-2},$$

where A and B are constants.

(a) Without evaluating any constants, state appropriate forms for expressing the following in partial fractions:

(i) $\dfrac{4x}{(x+4)(x^2+3)}$, [1]

(ii) $\dfrac{2x+1}{(x-2)(x+2)^2}$. [2]

(b) Show that $\displaystyle\int_3^4 \frac{3x}{(x+1)(x-2)}\,dx = \ln 5$. [6]

74. M/J 04/P3/Q5

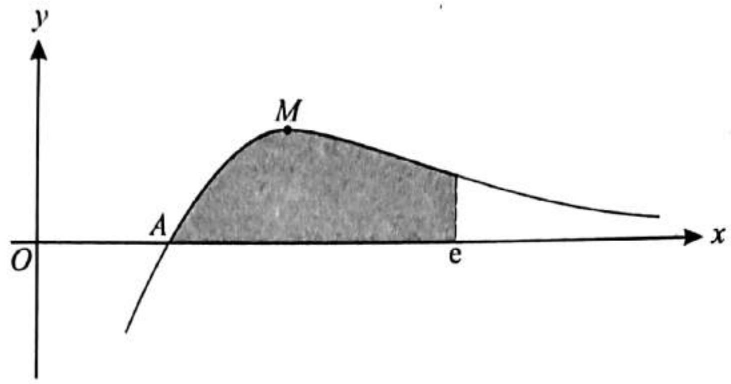
(i) Prove the identity

$$\sin^2\theta\cos^2\theta \equiv \tfrac{1}{8}(1 - \cos 4\theta).$$ [3]

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{3}\pi} \sin^2\theta\cos^2\theta\,d\theta.$$ [3]

75. M/J 04/P3/Q10



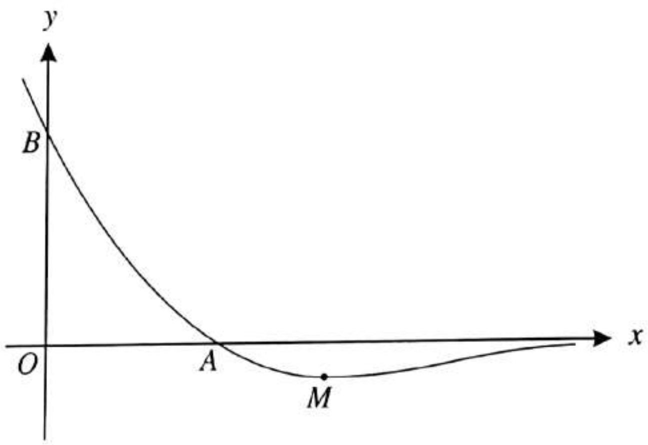
The diagram shows the curve $y = \dfrac{\ln x}{x^2}$ and its maximum point M. The curve cuts the x-axis at A.

(i) Write down the x-coordinate of A. [1]

(ii) Find the exact coordinates of M. [5]

(iii) Use integration by parts to find the exact area of the shaded region enclosed by the curve, the x-axis and the line $x = e$. [5]

76. O/N 03/P3/Q6



The diagram shows the curve $y = (3 - x)e^{-2x}$ and its minimum point M. The curve intersects the x-axis at A and the y-axis at B.

(i) Calculate the x-coordinate of M. [4]

(ii) Find the area of the region bounded by OA, OB and the curve, giving your answer in terms of e. [5]

77. O/N 03/P3/Q8

Let $f(x) = \dfrac{x^3 - x - 2}{(x-1)(x^2+1)}$.

(i) Express $f(x)$ in the form

$$A + \frac{B}{x-1} + \frac{Cx + D}{x^2+1},$$

where A, B, C and D are constants. [5]

(ii) Hence show that $\int_2^3 f(x)\,dx = 1$. [4]

78. M/J 03/P3/Q2

Find the exact value of $\int_0^1 xe^{2x}\,dx$. [4]

79. M/J 03/P3/Q10

(i) Prove the identity

$$\cot x - \cot 2x \equiv \operatorname{cosec} 2x.$$ [3]

(ii) Show that $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot x \, dx = \frac{1}{2} \ln 2.$ [3]

(iii) Find the exact value of $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \operatorname{cosec} 2x \, dx$, giving your answer in the form $a \ln b$. [4]

80. O/N 02/P3/Q2

Find the exact value of $\int_1^2 x \ln x \, dx$. [4]

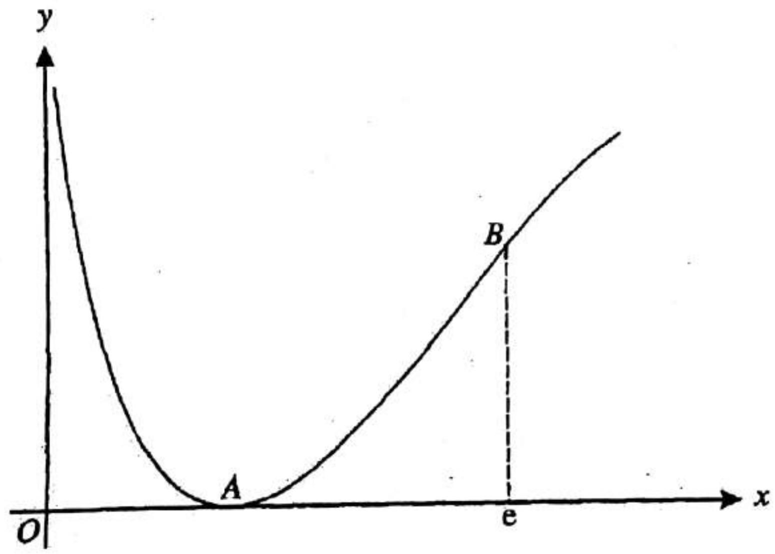
81. M/J 02/P3/Q6

Let $f(x) = \dfrac{4x}{(3x+1)(x+1)^2}$.

(i) Express f(x) in partial fractions. [5]

(ii) Hence show that $\int_0^1 f(x)\, dx = 1 - \ln 2$. [5]

82. M/J 02/P3/Q10



The function f is defined by $f(x) = (\ln x)^2$ for $x > 0$. The diagram shows a sketch of the graph of $y = f(x)$. The minimum point of the graph is A. The point B has x-coordinate e.

(i) State the x-coordinate of A. [1]

(ii) Show that $f''(x) = 0$ at B. [4]

(iii) Use the substitution $x = e^u$ to show that the area of the region bounded by the x-axis, the line x = e, and the part of the curve between A and B is given by

$$\int_0^1 u^2 e^u \, du.$$ [3]

(iv) Hence, or otherwise, find the exact value of this area. [3]

Answers Section

1. M/J 18/P32/Q4(ii)

(i) State integral of the form $a\ln(1+\cos x)$
Obtain integral $-\ln(1+\cos x)$
Substitute correct limits in correct order
Obtain answer $\ln\left(\frac{3}{2}\right)$, or equivalent **4**

2. M/J 18/P31/Q5

(i) State or imply $dx = -2\cos\theta\sin\theta\, d\theta$, or equivalent
Substitute for x and dx, and use Pythagoras
Obtain integrand $\pm 2\cos^2\theta$ **4**
Justify change of limits and obtain given answer correctly

(ii) Obtain indefinite integral of the form $a\theta + b\sin 2\theta$
Obtain $\theta + \frac{1}{2}\sin 2\theta$
Use correct limits correctly **4**
Obtain answer $\frac{1}{6}\pi$ with no errors seen

3. M/J 18/P33/Q3

Integrate by parts and reach $ax\sin 3x + b\int \sin 3x\,dx$
Obtain $\frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx$, or equivalent
Complete the integration and obtain $\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x$, or equivalent
Substitute limits correctly having integrated twice and obtained $ax\sin 3x + b\cos 3x$
Obtain answer $\frac{1}{18}(\pi - 2)$ OE **5**

4. M/J 18/P33/Q7(ii)

(i) State that the integrand is $3\sec^2(\theta-\alpha)$
State correct indefinite integral $3\tan(\theta-\alpha)$
Substitute limits correctly
Use $\tan(A \pm B)$ formula
Obtain the given exact answer correctly **5**

5. O/N 17/P32/Q9

(i) Integrate by parts and reach $ax^{\frac{3}{2}}\ln x + b\int x^{\frac{3}{2}}.\frac{1}{x}\,dx$
Obtain $\frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx$
Obtain integral $\frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}}$, or equivalent
Substitute limits correctly and equate to 2
Obtain the given answer correctly **5**

(ii) Evaluate a relevant expression or pair of expressions at $x = 2$ and $x = 4$
Complete the argument correctly with correct calculated values 2

(iii) Use the iterative formula correctly at least once
Obtain final answer 3.031
Show sufficient iterations to 5 d.p. to justify 3.031 to 3 d.p., or show there is a sign change in the interval (3.0305, 3.0315) 3

6. O/N 17/P31/Q8, O/N 17/P33/Q8

(i) Use a relevant method to determine a constant
Obtain one of the values $A = 2, B = 2, C = -1$
Obtain a second value
Obtain the third value 4

(ii) Integrate and obtain terms $2x + 2\ln(x+2) - \frac{1}{2}\ln(2x-1)$ (deduct **B1** for each error or omission) [The FT is on A, B and C]
Substitute limits correctly in an integral containing terms $a\ln(x+2)$ and $b\ln(2x-1)$, where $ab \neq 0$
Use at least one law of logarithms correctly
Obtain the given answer after full and correct working 5

7. M/J 17/P31/Q3(ii)

(i) State integral $k\ln(1+e^{-x})$ where $k = \pm 1$
State correct integral $-\ln(1+e^{-x})$
Use limits correctly
Obtain the **given answer** $\ln\left(\frac{2e}{e+1}\right)$ following full working 4

8. M/J 17/P31/Q9

(i) Carry out a relevant method to obtain A and B such that $\frac{1}{x(2x+3)} \equiv \frac{A}{x} + \frac{B}{2x+3}$, or equivalent
Obtain $A = \frac{1}{3}$ and $B = -\frac{2}{3}$, or equivalent 2

(ii) Separate variables and integrate one side
Obtain term $\ln y$
Integrate and obtain terms $\frac{1}{3}\ln x - \frac{1}{3}\ln(2x+3)$, or equivalent
Use $x = 1$ and $y = 1$ to evaluate a constant, or as limits, in a solution containing $a\ln y,\ b\ln x,\ c\ln(2x+3)$
Obtain correct solution in any form, e.g. $\ln y = \frac{1}{3}\ln x - \frac{1}{3}\ln(2x+3) + \frac{1}{3}\ln 5$
Obtain answer $y = 1.29$ (3s.f. only) 7

9. M/J 17/P33/Q4

Integrate by parts and reach $a\theta\cos\frac{1}{2}\theta + b\int\cos\frac{1}{2}\theta\, d\theta$
Complete integration and obtain indefinite integral $-2\theta\cos\frac{1}{2}\theta + 4\sin\frac{1}{2}\theta$
Substitute limits correctly, having integrated twice
Obtain final answer $(4-\pi)/\sqrt{2}$, or exact equivalent 4

10. M/J 17/P33/Q7(i,ii)

(i) Use correct quotient rule or product rule
Obtain correct derivative in any form
Equate derivative to zero and solve for x
Obtain $x = 2$ **4**

(ii) State or imply ordinates 1.6487…, 1.3591…, 1.4938…
Use correct formula, or equivalent, with $h = 1$ and three ordinates
Obtain answer 2.93 only **3**

11. M/J 17/P33/Q9(ii)

(i) Integrate and obtain terms $3\ln x = \dfrac{2}{x} - 2\ln(3x+2)$
[The FT is on A, B and C]
Note: Candidates who integrate the partial fraction $\dfrac{3x-2}{x^2}$ by parts should obtain

$3\ln x + \dfrac{2}{x} - 3$ or equivalent
Use limits correctly, having integrated all the partial fractions, in a solution containing terms $a\ln x + \frac{b}{x} + c\ln(3x+2)$
Obtain the given answer following full and exact working **5**

12. O/N 16/P32/Q5, O/N 16/P31/Q5

(i) *EITHER*: Use tan $2A$ formula to express LHS in terms of $\tan\theta$
Express as a single fraction in any correct form
Use Pythagoras or cos $2A$ formula
Obtain the given result correctly

OR: Express LHS in terms of $\sin 2\theta$, $\cos 2\theta$, $\sin\theta$ and $\cos\theta$
Express as a single fraction in any correct form
Use Pythagoras or cos $2A$ formula or $\sin(A-B)$ formula
Obtain the given result correctly **[4]**

(ii) Integrate and obtain a term of the form $a\ln(\cos 2\theta)$ or $b\ln(\cos\theta)$ (or secant equivalents)
Obtain integral $-\frac{1}{2}\ln(\cos 2\theta) + \ln(\cos\theta)$, or equivalent
Substitute limits correctly (expect to see use of both limits)
Obtain the given answer following full and correct working **[4]**

13. O/N 16/P33/Q6

(i) State or imply $du = \dfrac{1}{2\sqrt{x}}dx$
Substitute for x and dx throughout
Justify the change in limits and obtain the given answer **[3]**

(ii) Convert integrand into the form $A + \dfrac{B}{u+1}$
Obtain integrand $A = 1$, $B = -2$
Integrate and obtain $u - 2\ln(u+1)$
Substitute limits correctly in an integral containing terms au and $b\ln(u+1)$, where $ab \neq 0$
Obtain the given answer following full and correct working **[6]**
[The f.t. is on A and B.]

14. M/J 16/P32/Q3

Integrate by parts and reach $ax^2 \cos 2x + b\int x\cos 2x \, dx$

Obtain $-\frac{1}{2}x^2 \cos 2x + \int x \cos 2x$, or equivalent

Complete the integration and obtain $-\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x$, or equivalent

Use limits correctly having integrated twice

Obtain answer $\frac{1}{8}(\pi^2 - 4)$, or exact equivalent, with no errors seen

[5]

15. M/J 16/P32/Q7

(i) State or imply the form $A + \dfrac{B}{2x+1} + \dfrac{C}{x+2}$

State or obtain $A = 2$

Use a correct method for finding a constant

Obtain one of $B = 1, C = -2$

Obtain the other value

[5]

(ii) Integrate and obtain terms $2x + \frac{1}{2}\ln(2x+1) - 2\ln(x+2)$

Substitute correct limits correctly in an integral with terms $a\ln(2x+1)$ and $b\ln(x+2)$, where $ab \neq 0$

Obtain the given answer after full and correct working

[5]

16. M/J 16/P31/Q2

Integrate by parts and reach $axe^{-2x} + b\int e^{-2x} \, dx$

Obtain $-\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x} \, dx$, or equivalent

Complete the integration correctly, obtaining $-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$, or equivalent

Use limits $x = 0$ and $x = \frac{1}{2}$ correctly, having integrated twice

Obtain answer $\frac{1}{4} - \frac{1}{2}e^{-1}$, or exact equivalent

[5]

17. M/J 16/P33/Q7

(i) State or imply $du = 2x \, dx$, or equivalent

Substitute for x and dx throughout

Reduce to the given form and justify the change in limits

[3]

(ii) Convert integrand to a sum of integrable terms and attempt integration

Obtain integral $\frac{1}{2}\ln u + \dfrac{1}{u} - \dfrac{1}{4u^2}$, or equivalent

(deduct A1 for each error or omission)

Substitute limits in an integral containing two terms of the form $a\ln u$ and bu^{-2}

Obtain answer $\frac{1}{2}\ln 2 - \frac{5}{16}$, exact simplified equivalent

[5]

8. O/N 15/P32/Q10, O/N 15/P31/Q10

(i) Use the quotient rule

Obtain correct derivative in any form

Equate derivative to zero and solve for x

Obtain answer $x = \sqrt[3]{2}$, or exact equivalent

[4]

(ii) State or imply indefinite integral is of the form $k\ln(1 + x^3)$

State indefinite integral $\frac{1}{3}\ln(1 + x^3)$

Substitute limits correctly in an integral of the form $k\ln(1+x^3)$

State or imply that the area of R is equal to $\frac{1}{3}\ln(1+p^3)-\frac{1}{3}\ln 2$, or equivalent

Use a correct method for finding p from an equation of the form $\ln(1+p^3)=a$ or $\ln((1+p^3)/2)=b$

Obtain answer $p=3.40$ **[2]**

19. O/N 15/P33/Q5

State $du = 3\sin x\,dx$ or equivalent

Use identity $\sin 2x = 2\sin x\cos x$

Carry out complete substitution, for x and dx

Obtain $\int \frac{8-2u}{\sqrt{u}}\,du$, or equivalent

Integrate to obtain expression of form $au^{\frac{1}{2}}+bu^{\frac{3}{2}}$, $ab \neq 0$

Obtain correct $16u^{\frac{1}{2}}-\frac{4}{3}u^{\frac{3}{2}}$

Apply correct limits correctly

Obtain $\frac{20}{3}$ or exact equivalent **[8]**

20. O/N 15/P33/Q7

(i) Either Substitute $x=-1$ and evaluate

Obtain 0 and conclude $x+1$ is a factor

Or Divide by $x+1$ and obtain a constant remainder

Obtain remainder = 0 and conclude $x+1$ is a factor **[2]**

(ii) Attempt division, or equivalent, at least as far as quotient $4x^2+kx$

Obtain complete quotient $4x^2-5x-6$

State form $\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{4x+3}$

Use relevant method for finding at least one constant

Obtain one of $A=-2, B=1, C=8$

Obtain all three values

Integrate to obtain three terms each involving natural logarithm of linear form

Obtain $-2\ln(x+1)+\ln(x-2)+2\ln(4x+3)$, condoning no use of modulus signs and absence of $\ldots + c$ **[8]**

21. M/J 15/P32/Q6

(i) State or imply $du = -\frac{1}{2\sqrt{x}}dx$, or equivalent

Substitute for x and dx throughout

Obtain integrand $\frac{\pm 2(2-u)^2}{u}$, or equivalent

Show correct working to justify the change in limits and obtain the given answer with no errors seen **[4]**

(ii) Integrate and obtain at least two terms of the form $a\ln u$, bu, and cu^2

Obtain indefinite integral $8\ln u - 8u + u^2$, or equivalent

Substitute limits correctly

Obtain the given answer correctly having shown sufficient working **[4]**

22. M/J 15/P31/Q5

(a) Use identity $\tan^2 2x = \sec^2 2x - 1$
Obtain integral of form $ax + b\tan 2x$
Obtain correct $3x + \frac{1}{2}\tan 2x$, condoning absence of $+c$ [3]

(b) State $\sin x\cos\frac{1}{2}\pi + \cos x\sin\frac{1}{6}\pi$
Simplify integrand to $\cos\frac{1}{6}\pi + \frac{\cos x\sin\frac{1}{6}\pi}{\sin x}$ or equivalent
Integrate to obtain at least term of form $a\ln(\sin x)$
Apply limits and simplify to obtain two terms
Obtain $\frac{1}{8}\pi\sqrt{3} - \frac{1}{2}\ln(\frac{1}{\sqrt{2}})$ or equivalent [5]

23. M/J 15/P33/Q6

(i) Integrate and reach $\pm x\sin x \mp \int \sin x\,dx$
Obtain integral $x\sin x + \cos x$
Substitute limits correctly, must be seen since AG, and equate result to 0.5
Obtain the given form of the equation 4

(ii) *EITHER*: Consider the sign of a relevant expression at $a = 1$ and at another relevant value, e.g. $a = 1.5 \leqslant \frac{\pi}{2}$

OR: Using limits correctly, consider the sign of $[x\sin x + \cos x]_0^a - 0.5$, or compare the value of $[x\sin x + \cos x]_0^a$ with 0.5, for $a = 1$ AND for another relevant value, e.g $a = 1.5 \leqslant \frac{\pi}{2}$.

Complete the argument, so change of sign, or above and below stated, both with correct calculated values 2

(iii) Use the iterative formula correctly at least once
Obtain final answer 1.2461
Show sufficient iterations to 6 d.p. to justify 1.2461 to 4 d.p., or show there is a sign change in the interval (1.24605, 1.24615) 3

24. M/J 15/P33/Q10

(i) State or imply $f(x) \equiv \frac{A}{2x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$
Use a relevant method to determine a constant
Obtain one of the values $A = 2, B = -1, C = 3$
Obtain the remaining values A1 + 5

[Apply an analogous scheme to the form $\frac{A}{2x-1} + \frac{Dx+E}{(x+2)^2}$; the values being $A = 2$, $D = -1, E = 1$.]

(ii) Integrate and obtain terms $\frac{1}{2}\cdot 2\ln(2x-1) - \ln(x+2) - \frac{3}{x+2}$
Use limits correctly, namely substitution must be seen in at least two of the partial fractions to obtain M1 Integrate all 3 partial fractions and substitute in all three partial fractions for A1 since AG.

Obtain the given answer following full and exact working **5**
[The t marks are dependent on A, B, C etc.]
[SR: If B, C or E omitted, give B1M1 in part (i) and B1√B1√M1 in part (ii).]
[NB: Candidates who follow the A, D, E scheme in part (i) and then integrate $\frac{-x+1}{(x+2)^2}$ by parts should obtain $\frac{1}{2}\cdot 2\ln(2x-1)-\ln(x+2)+\frac{x-1}{x+2}$ (the third term is equivalent to $-\frac{3}{x+2}+1$).]

25. O/N 14/P32/Q2, O/N 14/P31/Q2

(i) State or imply ordinates 2, 1.1547…, 1, 1.1547…
Use correct formula, or equivalent, with $h=\frac{1}{6}\pi$ and four ordinates
Obtain answer 1.95 **[3]**

(ii) Make recognisable sketch of $y = \operatorname{cosec} x$ for the given interval
Justify a statement that the estimate will be an overestimate **[2]**

26. O/N 14/P32/Q6, O/N 14/P31/Q6

(i) Integrate and reach $bx\ln 2x - c\int x.\frac{1}{x}\,dx$, or equivalent
Obtain $x\ln 2x - \int x.\frac{1}{x}\,dx$, or equivalent
Obtain integral $x\ln 2x - x$, or equivalent
Substitute limits correctly and equate to 1, having integrated twice
Obtain a correct equation in any form, e.g. $a\ln 2a - a + 1 - \ln 2 = 1$
Obtain the given answer **[6]**

(ii) Use the iterative formula correctly at least once
Obtain final answer 1.94
Show sufficient iterations to 4 d.p. to justify 1.94 to 2d.p. or show that there is a sign change in the interval (1.935, 1.945). **[3]**

27. O/N 14/P33/Q6

(i) State or imply correct ordinates 1, 0.94259…, 0.79719…, 0.62000…
Use correct formula or equivalent with $h = 0.1$ and four y values
Obtain 0.255 with no errors seen **[3]**

(ii) Obtain or imply $a = -6$
Obtain x^4 term including correct attempt at coefficient
Obtain or imply $b = 27$

<u>Either</u> Integrate to obtain $x - 2x^3 + \frac{27}{5}x^5$, following their values of a and b
Obtain 0.259

<u>Or</u> Use correct trapezium rule with at least 3 ordinates
Obtain 0.259 (from 4) **[5]**

28. O/N 14/P33/Q10

State or imply $\frac{du}{dx} = e^x$
Substitute throughout for x and dx

Obtain $\int \frac{u}{u^2+3u+2}\,du$ or equivalent (ignoring limits so far)

State or imply partial fractions of form $\frac{A}{u+2}+\frac{B}{u+1}$, following their integrand

Carry out a correct process to find at least one constant for their integrand

Obtain correct $\frac{2}{u+2}-\frac{1}{u+1}$

Integrate to obtain $a\ln(u+2)+b\ln(u+1)$

Obtain $2\ln(u+2)-\ln(u+1)$ or equivalent, follow their A and B

Apply appropriate limits and use at least one logarithm property correctly

Obtain given answer $\ln\frac{8}{5}$ legitimately [10]

SR for integrand $\frac{u^2}{u(u+1)(u+2)}$

State or imply partial fractions of form $\frac{A}{u}+\frac{B}{u+1}+\frac{C}{u+2}$

Carry out a correct process to find at least one constant

Obtain correct $\frac{2}{u+2}-\frac{1}{u+1}$

...complete as above.

29. M/J 14/P31/Q2

State $\frac{du}{dx}=3\sec^2 x$ or equivalent

Express integral in terms of u and du (accept unsimplified and without limits)

Obtain $\int \frac{1}{3}u^{\frac{1}{2}}\,du$

Integrate $Cu^{\frac{1}{2}}$ to obtain $\frac{2C}{3}u^{\frac{3}{2}}$

Obtain $\frac{14}{9}$ [5]

30. M/J 14/P33/Q8

(i) Use a correct method for finding a constant
Obtain one of $A=3$, $B=3$, $C=0$
Obtain a second value
Obtain a third value 4

(ii) Integrate and obtain term $-3\ln(2-x)$

Integrate and obtain term of the form $k\ln(2+x^2)$

Obtain term $\frac{3}{2}\ln(2+x^2)$

Substitute limits correctly in an integral of the form $a\ln(2-x)+b\ln(2+x^2)$, where $ab\neq 0$
Obtain given answer after full and correct working 5

31. O/N 13/P32/Q3

EITHER: Integrate by parts and reach $kx^{\frac{1}{2}}\ln x - m\int x^{\frac{1}{2}}.\frac{1}{x}\,dx$

Obtain $2x^{\frac{1}{2}}\ln x - 2\int \frac{1}{x^{\frac{1}{2}}}\,dx$, or equivalent

Integrate again and obtain $2x^{\frac{1}{2}}\ln x - 4x^{\frac{1}{2}}$, or equivalent
Substitute limits $x = 1$ and $x = 4$, having integrated twice
Obtain answer $4(\ln 4 - 1)$, or exact equivalent

OR1: Using $u = \ln x$, or equivalent, integrate by parts and reach $kue^{\frac{1}{2}u} - m\int e^{\frac{1}{2}u}\,du$

Obtain $2ue^{\frac{1}{2}u} - 2\int e^{\frac{1}{2}u}\,du$, or equivalent

Integrate again and obtain $2ue^{\frac{1}{2}u} - 4e^{\frac{1}{2}u}$, or equivalent
Substitute limits $u = 0$ and $u = \ln 4$, having integrated twice
Obtain answer $4\ln 4 - 4$, or exact equivalent

OR2: Using $u = \sqrt{x}$, or equivalent, integrate and obtain $ku\ln u - m\int u.\frac{1}{u}\,du$

Obtain $4u\ln u - 4\int 1\,du$, or equivalent

Integrate again and obtain $4u\ln u - 4u$, or equivalent
Substitute limits $u = 1$ and $u = 2$, having integrated twice or quoted $\int \ln u\,du$ as $u\ln u \pm u$
Obtain answer $8\ln 2 - 4$, or exact equivalent

OR3: Integrate by parts and reach $I = \frac{x\ln x \pm x}{\sqrt{x}} + k\int \frac{x\ln x \pm x}{x\sqrt{x}}\,dx$

Obtain $I = \frac{x\ln x - x}{\sqrt{x}} + \frac{1}{2}I - \frac{1}{2}\int \frac{1}{\sqrt{x}}\,dx$

Integrate and obtain $I = 2\sqrt{x}\ln x - 4\sqrt{x}$, or equivalent
Substitute limits $x = 1$ and $x = 4$, having integrated twice
Obtain answer $4\ln 4 - 4$, or exact equivalent **[5]**

32. O/N 13/P32/Q5

(i) Use Pythagoras
Use the sin2*A* formula
Obtain the given result **[3]**

(ii) Integrate and obtain a $k\ln\sin\theta$ or $m\ln\cos\theta$ term, or obtain integral of the form $p\ln\tan\theta$

Obtain indefinite integral $\frac{1}{2}\ln\sin\theta - \frac{1}{2}\ln\cos\theta$, or equivalent, or $\frac{1}{2}\ln\tan\theta$

Substitute limits correctly
Obtain the given answer correctly having shown appropriate working **[4]**

33. O/N 13/P33/Q2

Carry out complete substitution including the use of $\frac{du}{dx} = 3$

Obtain $\int\left(\frac{1}{3} - \frac{1}{3u}\right)du$

Integrate to obtain form $k_1u + k_2\ln u$ or $k_1u + k_2\ln 3u$ where $k_1k_2 \neq 0$

Obtain $\frac{1}{3}(3x+1)-\frac{1}{3}\ln(3x+1)$ or equivalent, condoning absence of modulus signs and $+\ c$ [4]

34. O/N 13/P33/Q5

(i) Use integration by parts to obtain $axe^{-\frac{1}{2}x}+\int be^{-\frac{1}{2}x}\mathrm{d}x$

Obtain $-8xe^{-\frac{1}{2}x}+\int 8e^{-\frac{1}{2}x}\mathrm{d}x$ or unsimplified equivalent

Obtain $-8xe^{-\frac{1}{2}x}-16e^{-\frac{1}{2}x}$

Use limits correctly and equate to 9

Obtain given answer $p=2\ln\left(\frac{8p+16}{7}\right)$ correctly [5]

(ii) Use correct iteration formula correctly at least once

Obtain final answer 3.77

Show sufficient iterations to 5sf or better to justify accuracy 3.77 or show sign change in interval (3.765, 3.775) [3]

$[3.5\to 3.6766\to 3.7398\to 3.7619\to 3.7696\to 3.7723]$

35. M/J 13/P32/Q6

(i) Use correct quotient or chain rule to differentiate $\sec x$

Obtain given derivative, $\sec x\tan x$, correctly

Use chain rule to differentiate y

Obtain the given answer [4]

(ii) Using $\mathrm{d}x\sqrt{3}\sec^2\theta\,\mathrm{d}\theta$, or equivalent, express integral in terms of θ and $\mathrm{d}\theta$

Obtain $\int\sec\theta\,\mathrm{d}\theta$

Use limits $\frac{1}{6}\pi$ and $\frac{1}{3}\pi$ correctly in an integral form of the form $k\ln(\sec\theta+\tan\theta)$

Obtain a correct exact final answer in the given form, e.g. $\ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)$ [4]

36. M/J 13/P31/Q8

(a) Carry out integration by parts and reach $ax^2\ln x+b\int\frac{1}{2}x^2\mathrm{d}x$

Obtain $2x^2\ln x-\int\frac{1}{x}.2x^2\,\mathrm{d}x$

Obtain $2x^2\ln x-x^2$

Use limits, having integrated twice

Confirm given result $56\ln 2-12$ [5]

(b) State or imply $\frac{\mathrm{d}u}{\mathrm{d}x}=4\cos 4x$

Carry out complete substitution except limits

Obtain $\int(\frac{1}{4}-\frac{1}{4}u^2)\,\mathrm{d}u$ or equivalent

Integrate to obtain form $k_1u+k_2u^3$ with non-zero constants k_1, k_2 [5]

Use appropriate limits to obtain $\frac{11}{96}$

37. M/J 13/P31/Q9

(i) State or imply $R=5$ [3]

Use relevant trigonometry to find α

Obtain $\alpha=0.6435$

(ii) (a) Carry out appropriate method to find one value in given range
Obtain 1.80
Carry out appropriate method to find second value in given range
Obtain 5.77 and no other value **[4]**

(b) Express integrand as $k\sec^2(\theta - \text{their } \alpha)$ for any constant k
Integrate to obtain result $k\tan(\theta - \text{their } \alpha)$
Obtain correct answer $2\tan(\theta - 0.6435)$ **[3]**

38. M/J 13/P33/Q4

(i) State $R = 2$
Use trig formula to find α
Obtain $\alpha = \frac{1}{6}\pi$ with no errors seen **[3]**

(ii) Substitute denominator of integrand and state integral $k\tan(x - \alpha)$
State correct indefinite integral $\frac{1}{4}\tan\left(x - \frac{1}{6}\pi\right)$
Substitute limits
Obtain the given answer correctly **[4]**

39. O/N 12/P32/Q5, O/N 12/P31/Q5

(i) Use correct quotient or chain rule
Obtain the given answer correctly having shown sufficient working **[2]**

(ii) Use a valid method, e.g. multiply numerator and denominator by $\sec x + \tan x$, and a version of Pythagoras to justify the given identity **[1]**

(iii) Substitute, expand $(\sec x + \tan x)^2$ and use Pythagoras once
Obtain given identity **[2]**

(iv) Obtain integral $2\tan x - x + 2\sec x$
Use correct limits correctly in an expression of the form $a\tan x + bx + c\sec x$, or equivalent, where $abc \neq 0$
Obtain the given answer correctly **[3]**

40. O/N 12/P33/Q5

(i) Either Use correct product rule
Obtain $3e^{-2x} - 6xe^{-2x}$ or equivalent
Substitute $-\frac{1}{2}$ and obtain 6e

Or Take ln of both sides and use implicit differentiation correctly
Obtain $\frac{dy}{dx} = y\left(\frac{1}{x} - 2\right)$ or equivalent
Substitute $-\frac{1}{2}$ and obtain 6e **[3]**

(ii) Use integration by parts to reach $kxe^{-2x} \pm \int ke^{-2x}\,dx$
Obtain $-\frac{3}{2}xe^{-2x} + \int \frac{3}{2}e^{-2x}\,dx$ or equivalent
Obtain $-\frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x}$ or equivalent
Substitute correct limits correctly
Obtain $-\frac{3}{4}$ with no errors or inexact work seen **[5]**

41. O/N 12/P33/Q7

(i) State or imply $du = 2\cos 2x\ dx$ or equivalent
Express integrand in terms of u and du
Obtain $\int \frac{1}{2}u^3\left(1-u^2\right) du$ or equivalent
Integration to obtain an integral of the form $k_1u^4 + k_2u^6, k_1, k_2 \neq 0$
Use limits 0 and 1 or (if reverting to x) 0 and $\frac{1}{4}\pi$ correctly
Obtain $\frac{1}{24}$, or equivalent [6]

(ii) Use 40 and upper limit from part **(i)** in appropriate calculation
Obtain $k = 10$ with no errors seen [2]

42. M/J 12/P32/Q8

(i) State or imply $2u\ du = -dx$, or equivalent
Substitute for x and dx throughout
Obtain integrand $\frac{-10u}{6-u^2+u}$, or equivalent
Show correct working to justify the change in limits and obtain the given answer correctly [4]

(ii) State or imply the form of fractions $\frac{A}{3-u} + \frac{B}{2+u}$ and use a relevant method to find A or B
Obtain $A = 6$ and $B = -4$
Integrate and obtain $-6\ln(3-u) - 4\ln(2+u)$, or equivalent
Substitute limits correctly in an integral of the form $a\ln(3-u) + b\ln(2+u)$
Obtain the given answer correctly having shown sufficient working [6]
[The f.t. is on A and B.]

43. M/J 12/P31/Q9

State or imply form $A + \frac{B}{2x+1} + \frac{C}{x+2}$
State or obtain $A = 2$
Use correct method for finding B or C
Obtain $B = 1$
Obtain $C = -3$
Obtain $2x + \frac{1}{2}\ln(2x+1) - 3\ln(x+2)$ [Deduct B1√ for each error or omission]
Substitute limits in expression containing $a\ln(2x+1) + b\ln(x+2)$
Show full and exact working to confirm that $8 + \frac{1}{2}\ln 9 - 3\ln 6 + 3\ln 2$, or an equivalent expression, simplifies to given result $8 - \ln 9$ [10]
[SR: If A omitted from the form of fractions, give B0B0M1A0A0 in (i); B0√B1√B1√M1A0 in (ii).]
[SR: For a solution starting with $\frac{M}{2x+1} + \frac{Nx}{x+2}$ or $\frac{Px}{2x+1} + \frac{Q}{x+2}$, give B0B0M1A0A0 in (i); B1√B1√B1√, if recover correct form, M1A0 in (ii).]
[SR: For a solution starting with $\frac{B}{2x+1} + \frac{Dx+E}{x+2}$, give M1A1 for one of $B = 1$, $D = 2$, $E = 1$

and A1 for the other two constants; then give B1B1 for $A = 2$, $C = -3$.]

[SR: For a solution starting with $\frac{Fx+G}{2x+1}+\frac{C}{x+2}$, give M1A1 for one of $C = -3$, $F = 4$, $G = 3$ and A1 for the other constants or constant; then give B1B1 for $A = 2$, $B = 1$.]

44. M/J 12/P33/Q8

(i) State or imply the form $A+\frac{B}{x+1}+\frac{C}{2x-3}$

State or obtain $A = 2$

Use a correct method for finding a constant

Obtain $B = -2$

Obtain $C = -1$ **[5]**

(ii) Obtain integral $2x-2\ln(x+1)-\frac{1}{2}\ln(2x-3)$

(Deduct B1√ for each error or omission. The f.t. is on A, B, C.)

Substitute limits correctly in an expression containing terms $a\ln(x + 1)$ and $b\ln(2x - 3)$

Obtain the given answer following full and exact working **[5]**

[SR: If A omitted from the form of fractions, give B0B0M1A0A0 in (i); B1 B1√M1A0 in (ii).]

[SR: For a solution starting with $\frac{B}{x+1}+\frac{Dx+E}{2x-3}$, give M1A1 for one of $B = -2$, $D = 4$, $E = -7$ and A1 for the other two constants; then give B1B1 for $A = 2$, $C = -1$.]

[SR: For a solution starting with $\frac{Fx+G}{x+1}+\frac{C}{2x-3}$ or with $\frac{Fx}{x+1}+\frac{C}{2x-3}$, give M1A1 for one of $C = -1$, $F = 2$, $G = 0$ and A1 for the other constants or constant; then give B1B1 for $A = 2$, $B = -2$.]

45. O/N 11/P32/Q8, O/N 11/P31/Q8

(i) Use any relevant method to determine a constant

Obtain one of the values $A = 3$, $B = 4$, $C = 0$

Obtain a second value

Obtain the third value **[4]**

(ii) Integrate and obtain term $-3\ln(2-x)$

Integrate and obtain term $k\ln(4+x^2)$

Obtain term $2\ln(4+x^2)$

Substitute correct limits correctly in a complete integral of the form $a\ln(2-x)+b\ln(4+x^2)$, $ab \neq 0$

Obtain given answer following full and correct working **[5]**

46. O/N 11/P33/Q10

(i) State or imply $\frac{du}{dx}=\sec^2 x$

Express integrand in terms of u and du

Integrate to obtain $\frac{u^{n+1}}{n+1}$ or equivalent

Substitute correct limits correctly to confirm given result $\frac{1}{n+1}$ **[4]**

(ii) (a) Use $\sec^2 x = 1+\tan^2 x$ twice

Obtain integrand $\tan^4 x+\tan^2 x$

Apply result from part (i) to obtain $\frac{1}{3}$ **[3]**

Or

Use $\sec^2 x = 1 + \tan^2 x$ and the substitution from **(i)**

Obtain $\int u^2 du$

Apply limits correctly and obtain $\frac{1}{3}$

(b) Arrange, perhaps implied, integrand to
$t^9 + t^7 + 4(t^7 + t^5) + t^5 + t^3$
Attempt application of result from part **(i)** at least twice

Obtain $\frac{1}{8}+\frac{4}{6}+\frac{1}{4}$ and hence $\frac{25}{24}$ or exact equivalent **[3]**

47. M/J 11/P31/Q7

(i) State or imply $dx = 2t\,dt$ or equivalent
Express the integral in terms of x and dx

Obtain given answer $\int_1^5 (2x-2)\ln x\, dx$, including change of limits **[3]**

(ii) Attempt integration by parts obtaining $(ax^2 + bx)\ln x \pm \int (ax^2 + bx)\frac{1}{x}dx$ or equivalent

Obtain $(x^2 - 2x)\ln x - \int (x^2 - 2x)\frac{1}{x}dx$ or equivalent

Obtain $(x^2 - 2x)\ln x - \frac{1}{2}x^2 + 2x$

Use limits correctly having integrated twice
Obtain 15 ln 5 – 4 or exact equivalent **[5]**
[Equivalent for M1 is $(2x-2)(ax\ln x + bx) - \int (ax\ln x + bx)\,2dx$]

48. M/J 11/P31/Q9

(i) Express $\cos 4\theta$ as $2\cos^2 2\theta - 1$ or $\cos^2 2\theta - \sin^2 2\theta$ or $1 - 2\sin^2 2\theta$
Express $\cos 4\theta$ in terms of $\cos\theta$
Obtain $8\cos^4\theta - 8\cos^2\theta + 1$
Use $\cos 2\theta = 2\cos^2\theta - 1$ to obtain given answer $8\cos^4\theta - 3$ **[4]**

(ii) (a) State or imply $\cos^4\theta = \frac{1}{2}$
Obtain 0.572
Obtain –0.572 **[3]**

(b) Integrate and obtain form $k_1\theta + k_2\sin 4\theta + k_3\sin 2\theta$
Obtain $\frac{3}{8}\theta + \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta$
Obtain $\frac{3}{32}\pi + \frac{1}{4}$ following completely correct work **[3]**

49. M/J 11/P33/Q3

Attempt integration by parts and reach $k(1-x)e^{-\frac{1}{2}x} \pm k\int e^{-\frac{1}{2}x}dx$, or equivalent

Obtain $-2(1-x)e^{-\frac{1}{2}x} - 2\int e^{-\frac{1}{2}x}dx$, or equivalent

Integrate and obtain $-2(1-x)e^{-\frac{1}{2}x} + 4e^{-\frac{1}{2}x}$, or equivalent
Use limits $x = 0$ and $x = 1$, having integrated twice
Obtain the given answer correctly **[5]**

50. O/N 10/P32/Q5, O/N 10/P31/Q5

(i) State or imply $dx = 2\cos\theta\, d\theta$, or $\frac{dx}{d\theta} = 2\cos\theta$, or equivalent
Substitute for x and dx throughout the integral
Obtain the given answer correctly, having changed limits and shown sufficient working **[3]**

(ii) Replace integrand by $2 - 2\cos 2\theta$, or equivalent
Obtain integral $2\theta - \sin 2\theta$, or equivalent
Substitute limits correctly in an integral of the form $a\theta \pm b\sin 2\theta$, where ab þ 0
Obtain answer $\frac{1}{3}\pi - \frac{\sqrt{3}}{2}$ or exact equivalent **[4]**
[The f.t. is on integrands of the form $a + c\cos 2\theta$, where ac þ 0.]

51. O/N 10/P33/Q4

(i) Obtain derivative of form $k\cos 3x \sin 3x$, any constant k
Obtain $-24\cos 3x \sin 3x$ or unsimplified equivalent
Obtain $-6\sqrt{3}$ or exact equivalent **[3]**

(ii) Express integrand in the form $a + b\cos 6x$, where $ab \neq 0$
Obtain $2 + 2\cos 6x$ o.e.
Obtain $2x + \frac{1}{3}\sin 6x$ or equivalent, condoning absence of $+ c$, ft on a, b **[3]**

52. O/N 10/P33/Q5

State or imply form $\frac{A}{2x+1} + \frac{B}{x+2}$
Use relevant method to find A or B
Obtain $\frac{4}{2x+1} - \frac{1}{x+2}$
Integrate and obtain $2\ln(2x+1) - \ln(x+2)$ (ft on their A, B)
Apply limits to integral containing terms $a\ln(2x+1)$ and $b\ln(x+2)$ and apply a law of logarithms correctly.
Obtain given answer $\ln 50$ correctly **[7]**

53. M/J 10/P32/Q2

Integrate by parts and reach $\pm x^2\cos x \pm \int 2x\cos x\, dx$
Obtain $-x^2\cos x + \int 2x\cos x\, dx$, or equivalent
Complete the integration, obtaining $-x^2\cos x + 2x\sin x + 2\cos x$, or equivalent
Substitute limits correctly, having integrated twice
Obtain the given answer correctly **[5]**

54. M/J 10/P32/Q10

(i) *EITHER*: Divide by denominator and obtain quadratic remainder
Obtain $A = 1$
Use any relevant method to obtain B, C or D
Obtain one correct answer
Obtain $B = 2$, $C = 1$ and $D = -3$

OR: Reduce RHS to a single fraction and equate numerators, or equivalent
Obtain $A = 1$

Use any relevant method to obtain B, C or D
Obtain one correct answer
Obtain $B = 2$, $C = 1$ and $D = -3$
[SR: If $A = 1$ stated without working give B1.] [5]

(ii) Integrate and obtain $x + 2\ln x - \frac{1}{x} - \frac{3}{2}\ln(2x-1)$, or equivalent
(The f.t. is on A, B, C, D. Give B2√ if only one error in integration; B1√ if two.)
Substitute limits correctly in the complete integral
Obtain given answer correctly following full and exact working [5]

55. M/J 10/P31/Q4

(i) State correct expansion of $\cos(3x - x)$ or $\cos(3x + x)$
Substitute expansions in $\frac{1}{2}(\cos 2x - \cos 4x)$, or equivalent
Simplify and obtain the given identity correctly [3]

(ii) Obtain integral $\frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x$
Substitute limits correctly in an integral of the form $a\sin 2x + b\sin 4x$
Obtain given answer following full, correct and exact working [3]

56. M/J 10/P31/Q8

(i) State or imply the form $\frac{A}{x+1} + \frac{B}{x+3}$ and use a relevant method to find A or B
Obtain $A = 1, B = -1$ [2]

(ii) Square the result of part (i) and substitute the fractions of part (i)
Obtain the given answer correctly [2]

(iii) Integrate and obtain $-\frac{1}{x+1} - \ln(x+1) + \ln(x+3) - \frac{1}{x+3}$
Substitute limits correctly in an integral containing at least two terms of the correct form
Obtain given answer following full and exact working [5]

57. M/J 10/P33/Q7

(i) Use correct $\cos(A + B)$ formula to express $\cos 3\theta$ in terms of trig functions of 2θ and θ
Use correct trig formulae and Pythagoras to express $\cos 3\theta$ in terms of $\cos\theta$
Obtain a correct expression in terms of $\cos\theta$ in any form
Obtain the given identity correctly [4]
[SR: Give M1 for using correct formulae to express RHS in terms of $\cos\theta$ and $\cos 2\theta$, then M1A1 for expressing in terms of either only $\cos 3\theta$ and $\cos\theta$, or only $\cos 2\theta$, $\sin 2\theta$, $\cos\theta$, and $\sin\theta$, and A1 for obtaining the given identity correctly.]

(ii) Use identity and integrate, obtaining terms $\frac{1}{4}(\frac{1}{3}\sin 3\theta)$ and $\frac{1}{4}(3\sin\theta)$, or equivalent
Use limits correctly in an integral of the form $k\sin 3\theta + l\sin\theta$
Obtain answer $\frac{2}{3} - \frac{3}{8}\sqrt{3}$, or any exact equivalent [4]

58. O/N 09/P32/Q6

(i) State or imply $\frac{dx}{d\theta} = 2\sec^2\theta$ or $dx = 2\sec^2\theta\, d\theta$

Substitute for x and dx throughout
Obtain any correct form in terms of θ
Obtain the given form correctly (including the limits) **[4]**

(ii) Use cos $2A$ formula, replacing integrand by $a + b\cos 2\theta$, where $ab \neq 0$
Integrate and obtain $\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$
Use limits $\theta = 0$ and $\theta = \frac{1}{4}\pi$
Obtain answer $\frac{1}{8}(\pi + 2)$, or exact quivalent **[4]**

59. O/N 09/P31/Q5

(i) *EITHER*: Use double angle formulae correctly to express LHS in terms of trig functions of 2θ
Use trig formulae correctly to express LHS in terms of $\sin\theta$, converting at least two terms
Obtain expression in any correct form in terms of $\sin\theta$
Obtain given answer correctly

OR: Use double angle formulae correctly to express RHS in terms of trig functions of 2θ
Use trig formulae correctly to express RHS in terms of $\cos 4\theta$ and $\cos 2\theta$
Obtain expression in any correct form in terms of $\cos 4\theta$ and $\cos 2\theta$
Obtain given answer correctly **[4]**

(ii) State indefinite integral $\frac{1}{4}\sin 4\theta - \frac{4}{2}\sin 2\theta + 3\theta$, or equivalent
(award B1 if there is just one incorrect term)
Use limits correctly, having attempted to use the identity
Obtain answer $\frac{1}{32}(2\pi - \sqrt{3})$, or any simplified exact equivalent **[4]**

60. M/J 09/P3/Q10

(i) **EITHER** Use product and chain rule
Obtain correct derivative in any form

OR Square and differentiate LHS by chain rule and RHS by product rule or as powers
Obtain correct result in any form

Set $\frac{dy}{dx}$ equal to zero and make reasonable attempt to solve for $x \neq 0$
Obtain answer $x = \sqrt{\frac{2}{3}}$, or exact equivalent, correctly **4**

(ii) State or imply $dx = \cos\theta\, d\theta$ or $\frac{dx}{d\theta} = \cos\theta$
Substitute for x and dx throughout the integral $\int y\,dx$
Obtain the given form correctly with no errors seen **3**

(iii) Attempt integration and reach indefinite integral of the form $a\theta + b\sin 4\theta$, where $ab \neq 0$
Obtain indefinite integral $\frac{1}{8}\theta - \frac{1}{32}\sin 4\theta$, or equivalent
Substitute limits correctly
Obtain exact answer $\frac{1}{16}\pi$ **4**

[Working to carry out the change of limits is needed for the A mark in **(ii)** but, if omitted, can be earned retrospectively if it is seen in part **(iii)**]

61. O/N 08/P3/Q9

(i) Integrate by parts and reach $kxe^{\frac{1}{2}x} - k\int e^{\frac{1}{2}x}\,dx$

Obtain $2xe^{\frac{1}{2}x} - 2\int e^{\frac{1}{2}x}\,dx$

Complete the integration, obtaining $2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x}$, or equivalent
Substitute limits correctly and equate result to 6, having integrated twice

Rearrange and obtain $a = e^{-\frac{1}{2}a} + 2$ [5]

(ii) Make recognizable sketch of a relevant exponential graph, e.g. $y = e^{-\frac{1}{2}x} + 2$
Sketch a second relevant straight line graph, e.g. $y = x$, or curve, and indicate the root

(iii) Consider sign of $x - e^{-\frac{1}{2}x} - 2$ at $x = 2$ and $x = 2.5$, or equivalent
Justify the given statement with correct calculations and argument [2]

(iv) Use the iterative formula $x_{n+1} = 2 + e^{-\frac{1}{2}x_n}$ correctly at least once, with $2 \le x_n \le 2.5$
Obtain final answer 2.31
Show sufficient iterations to at least 4 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in the interval (2.305, 2.315) [3]

62. M/J 08/P3/Q7

(i) State or imply the form $A + \dfrac{B}{x+1} + \dfrac{C}{x+3}$

State or obtain $A = 1$
Use correct method for finding B or C
Obtain $B = \frac{1}{2}$

Obtain $C = -\frac{3}{2}$ [5]

(ii) Obtain integral $x + \frac{1}{2}\ln(x+1) - \frac{3}{2}\ln(x+3)$
[Award B1√ if only one error. The f.t. is on A, B, C.]
Substitute limits correctly
Obtain given answer following full and exact working [4]
[SR: if A omitted, only M1 in part **(i)** is available, then in part **(ii)** B1√ for each correct integral and M1.]

63. O/N 07/P3/Q1

Obtain indefinite integral of the form $a\ln(2x-1)$, where $a = \frac{1}{2}$, 1, or 2

Use limits and obtain equation $\frac{1}{2}\ln(2k-1) = 1$

Use correct method for solving an equation of the form $a\ln(2k-1) = 1$, where $a = \frac{1}{2}$, 1, or 2, for k

Obtain answer $k = \frac{1}{2}(e^2 + 1)$, or exact equivalent [4]

64. O/N 07/P3/Q3

Using 1 and $\ln x$ as parts reach $x\ln x \pm \int x.\dfrac{1}{x}\,dx$

Obtain indefinite integral $x\ln x - x$
Substitute correct limits correctly
Obtain given answer [4]

65. M/J 07/P3/Q5

(i) State answer $R = 2$
Use trig formula to find a

Obtain $a = \frac{1}{3}\pi$, or 60°

[For the M1 condone a sign error in the expansion of $\cos(\theta - a)$, but the subsequent trigonometric work must be correct.]

[SR: The answer $a = \tan^{-1}\left(\sqrt{3}\right)$ earns M1 only.]

(ii) State that the integrand is of the form $a\sec^2(\theta - a)$

State correct indefinite integral $\frac{1}{4}\tan\left(\theta - \frac{1}{3}\pi\right)$

Use limits correctly in an integral of the form $a\tan(\theta - a)$

Obtain given answer correctly following full and exact working

[The f.t. is on R and a.] **4**

66. M/J 07/P3/Q7

(i) State or imply $du = \frac{1}{2\sqrt{x}}\,dx$, or $2u\,du = dx$, or $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, or equivalent

Substitute for x and dx throughout the integral

Obtain the given form of indefinite integral correctly with not errors seen **3**

(ii) Attempting to express the integrand as $\frac{A}{u} + \frac{B}{4-u}$, use a correct method to find either A or B

Obtain $A = \frac{1}{2}$ and $B = \frac{1}{2}$

Integrate and obtain $\frac{1}{2}\ln u - \frac{1}{2}\ln(4-u)$, or equivalent

Use limits $u = 1$ and $u = 2$ correctly, or equivalent, in an integral of the form $c\ln u + d\ln(4-u)$

Obtain given answer correctly following full and exact working **6**

67. O/N 06/P3/Q8

(i) *EITHER*: State or imply $f(x) = \frac{A}{2x+1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

Use any relevant method to obtain a constant

Obtain one of the values $A = 2$, $B = -1$, $C = 3$

Obtain the remaining two values

[A correct solution starting with third term $\frac{Cx}{(x+1)^2}$ or $\frac{Cx+D}{(x+1)^2}$ is also possible.]

OR: State or imply $f(x) = \frac{A}{2x+1} + \frac{Dx+E}{(x+1)^2}$

Use any relevant method to obtain a constant

Obtain one of the values $A = 2$, $D = -1$, $E = 2$

Obtain the remaining two values **5**

(ii) Integrate and obtain terms $\frac{1}{2}.2\ln(2x+1) - \ln(x+1) - \frac{3}{x+1}$, or equivalent

Use limits correctly, having integrated all the partial fractions

Obtain given answer following full and exact working **5**

[The f.t. is on A, B, C etc.]

[SR: If B, C, or E are omitted, give B1M1 in part **(i)** and B1 B1 M1 in part **(ii)**: max 5/10.]

68. M/J 06/P3/Q8

(i) Use product rule

Obtain derivative in any correct form e.g. $\frac{x^{\frac{1}{2}}}{x} + \frac{x^{-\frac{1}{2}}}{x}$, In x

Equate derivative to zero and solve for In x

Obtain $x = e^{-2}$ (or $\frac{1}{e^2}$) or equivalent 4

(ii) *EITHER*: Attempt integration by parts with u – In x

Obtain $\frac{2}{3}x^{\frac{3}{2}}\ \text{In}\ x - \int \frac{2}{3}x^{\frac{3}{2}}.\ \frac{1}{x}\text{d}x$, or equivalent

OR: Attempt integration by parts with $u = x^{\frac{1}{2}}$

Obtain $x^{\frac{1}{2}}(x\ \text{In}\ x - x) - \int (x\text{In}\ x - x), \frac{x^{-\frac{1}{2}}}{2}\ \text{d}x$

Obtain indefinite integral $\frac{2}{3}x^{\frac{3}{2}}\ \text{In}\ x - \frac{4}{9}x^{\frac{3}{2}}$, or equivalent

Use $x = 1$ and $x = 4$ as limits

Obtain answer 4.28 5

69. O/N 05/P3/Q6

(i) State $\frac{\text{d}x}{\text{d}\theta} = 2\sin\theta\cos\theta$, or $\text{d}x = 2\sin\theta\cos\theta\,\text{d}\theta$

Substitute for x and dx throughout

Obtain any correct form in terms of θ

Reduce to the given form correctly [4]

(ii) Use cos $2A$ formula, replacing integrand by $a + b\cos 2\theta$, where $ab \neq 0$

Integrate and obtain $\theta - \frac{1}{2}\sin 2\theta$

Use limits $\theta = 0$ and $\theta = \frac{1}{6}\pi$

Obtain exact answer $\frac{1}{6}\pi - \frac{1}{4}\sqrt{3}$, or equivalent [4]

70. M/J 05/P3/Q4

(i) State or imply $\text{d}x = \sec^2\theta\,\text{d}\theta$ or $\frac{\text{d}x}{\text{d}\theta} = \sec^2\theta$

Substitute for x and dx throughout the integral

Obtain integral in terms of θ in any correct form

Reduce to the given form correctly 4

(ii) State integral $\frac{1}{2}\sin 2\theta$

Use limits $\theta = 0$ and $\theta = \frac{1}{4}\pi$ correctly in integral of the form $k\sin 2\theta$

Obtain answer ½ or 0.5 3

71. M/J 05/P3/Q9

(i) Use quotient or product rule

Obtain derivative in any correct form

Equate derivative to zero and solve for x or x^2

Obtain $x = 1$ correctly 4

[Differentiating $(x^2 + 1)y = x$ using the product rule can also earn the first **M1A1.**]

[SR: if the quotient rule is misused, with a 'reversed' numerator or v instead of v^2 in the denominator, award **M0A0** but allow the following **M1A1**.]

(ii) Obtain indefinite integral of the form $k\ln(x^2+1)$, where k = ½, 1 **or** 2
Use limits $x = 0$ and $x = p$ correctly, or equivalent
Obtain answer ½ $\ln(p^2+1)$ **3**
[Also accept $-\ln\cos\theta$ or $\ln\cos\theta$, where $x = \tan\theta$, for the first **M1***.]

(iii) Equate to 1 and convert equation to the form $p^2+1=\exp(1/k)$
Obtain answer $p = 2.53$ **2**

72. O/N 04/P3/Q7

(i) Use product or quotient rule
Obtain first derivative $2xe^{-\frac{1}{2}x}-\frac{1}{2}x^2e^{-\frac{1}{2}x}$ or equivalent
Equate derivative to zero and solve for non-zero x
Obtain answer $x = 4$ **4**

(ii) Integrate by parts once, obtaining $kx^2e^{-\frac{1}{2}x}+l\int xe^{-\frac{1}{2}x}dx$, where $kl\neq 0$
Obtain integral $-2x^2e^{-\frac{1}{2}x}+4\int xe^{-\frac{1}{2}x}dx$, or any unsimplified equivalent
Complete the integration, obtaining $-2(x^2+4x+8)e^{-\frac{1}{2}x}$ or equivalent
Having integrated by parts twice, use limits $x = 0$ and $x = 1$ in the complete integral
Obtain simplified answer $16-26e^{-\frac{1}{2}}$ or equivalent **5**

73. O/N 04/P3/Q8

(a) (i) State answer $\frac{A}{x+4}+\frac{Bx+C}{x^2+3}$ **1**

(ii) State answer $\frac{A}{x-2}+\frac{Bx+C}{(x+2)^2}$ or $\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$ **2**

[Award B1 if the B term is omitted or for the form $\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{(x+2)^2}$.]

(b) Stating or implying $f(x)\equiv\frac{A}{x+1}+\frac{B}{x-2}$, use a relevant method to determine A or B
Obtain $A = 1$ and $B = 2$
[SR: If $A = 1$ and $B = 2$ stated without working, award B1 + B1.]
Integrate and obtain terms $\ln(x+1)+2\ln(x-2)$
Use correct limits correctly in the complete integral
Obtain given answer ln 5 following full and exact working **6**

74. M/J 04/P3/Q5

(i) Make relevant use of formula for sin 2θ or cos 2θ
Make relevant use of formula for cos 4θ
Complete proof of the given result **[3]**

(ii) Integrate and obtain $\frac{1}{8}(\theta - \frac{1}{4}\sin 4\theta)$ or equivalent
Use limits correctly with an integral of the form $a\theta + b\sin 4\theta$, where $ab \neq 0$
Obtain answer $\frac{1}{8}(\frac{1}{3}\pi + \frac{\sqrt{3}}{8})$, or exact equivalent [3]

75. M/J 04/P3/Q10

(i) State x-coordinate of A is 1 1
(ii) Use product or quotient rule
Obtain derivative in any correct form e.g. $-\frac{2\ln x}{x^3} + \frac{1}{x}.\frac{1}{x^2}$
Equate derivative to zero and solve for $\ln x$
Obtain $x = e^{\frac{1}{2}}$ or equivalent (accept 1.65)
Obtain $y = \frac{1}{2e}$ or exact equivalent not involving ln 5
[SR: if the quotient rule is misused, with a 'reversed' numerator or x^2 instead of x^4 in the denominator, award M0A0 but allow the following M1A1A1.]
(iii) Attempt integration by parts, going the correct way
Obtain $-\frac{\ln x}{x} + \int \frac{1}{x}.\frac{1}{x}\,dx$ or equivalent
Obtain indefinite integral $-\frac{\ln x}{x} - \frac{1}{x}$
Use x-coordinate of A and e as limits, having integrated twice
Obtain exact answer $1 - \frac{2}{e}$, or equivalent 5
[If $u = \ln x$ is used, apply an analogous scheme to the result of the substitution.]

76. O/N 03/P3/Q6

(i) Use product or quotient rule to find derivative
Obtain derivative in any correct form
Equate derivative to zero and solve a linear equation in x
Obtain answer $3\frac{1}{2}$ only [4]

(ii) State first step of the form $\pm\frac{1}{2}(3-x)e^{-2x} \pm \frac{1}{2}\int e^{-2x}dx$, with or without 3
State correct first step e.g. $-\frac{1}{2}(3-x)e^{-2x} - \frac{1}{2}\int e^{-2x}dx$, or equivalent, with or without 3
Complete the integration correctly obtaining $-\frac{1}{2}(3-x)e^{-2x} + \frac{1}{4}e^{-2x}$, or equivalent
Substitute limits $x = 0$ and $x = 3$ correctly in the complete integral
Obtain answer $\frac{1}{4}(5 + e^{-6})$, or exact equivalent (allow e^0 in place of 1) [5]

77. O/N 03/P3/Q8

(i) *EITHER*: Divide by denominator and obtain a quadratic remainder
Obtain $A = 1$
Use any relevant method to obtain B, C or D
Obtain one correct answer
Obtain $B = -1$, $C = 2$, $D = 0$

OR: Reduce *RHS* to a single fraction and identify numerator with that of f(x)
Obtain $A = 1$
Use any relevant method to obtain B, C or D
Obtain one correct answer
Obtain $B = -1$, $C = 2$, $D = 0$ [5]

(ii) Integrate and obtain terms $x - \ln(x-1)$, or equivalent
Obtain third term $\ln(x^2+1)$, or equivalent
Substitute correct limits correctly in the complete integral
Obtain given answer following full and exact working **[4]**

[If $B = 0$ the first B1√ is not available.]
[SR: If A is omitted in part (i), treat as if $A = 0$. Thus only M1M1 and B1√B1√M1 are available.]

78. M/J 03/P3/Q2

State first step of the form $kxe^{2x} \pm \int ke^{2x}\,dx$
Complete the first step correctly
Substitute limits correctly having attempted the further integration of ke^{2x}
Obtain answer ¼ $(e^2 + 1)$ or exact equivalent of the form $ae^2 + b$, having used $e^0 = 1$ throughout **[4]**

79. M/J 03/P3/Q10

(i) *EITHER* Make relevant use of the correct sin 2*A* formula
Make relevant use of the correct cos 2*A* formula
Derive the given result correctly

OR Make relevant use of the tan 2*A* formula
Make relevant use of $1 + \tan^2 A = \sec^2 A$ or $\cos^2 A + \sin^2 A = 1$
Derive the given result correctly **[3]**

(ii) State or imply indefinite integral is ln sin *x*, or equivalent
Substitute correct limits correctly
Obtain given exact answer correctly **[3]**

(iii) *EITHER* State indefinite integral of cos 2*x* is of the form *k* ln sin 2*x*
State correct integral ½ ln sin 2*x*
Substitute limits correctly throughout
Obtain answer ¼ 1n 3, or equivalent

OR State or obtain indefinite integral of cosec 2*x* is of the form *k* ln tan *x*, or equivalent
State correct integral ½ ln tan *x*, or equivalent
Substitute limits correctly
Obtain answer ¼ ln 3, or equivalent **[4]**

80. O/N 02/P3/Q2

EITHER: State first step of the form $kx^2 \ln x \pm \int kx^2 \cdot \frac{1}{x}\,dx$

Obtain correct first step i.e. $\frac{1}{2}x^2 \ln x - \int \frac{1}{2}x\,dx$

Complete a second integration and substitute both limits correctly

Obtain correct answer $2\ln 2 - \frac{3}{4}$, or exact two-term equivalent

OR: State first step of the form $I = x(x\ln x \pm x) \pm \int (x\ln x \pm x)\,dx$

Obtain correct first step i.e. $I = x(x\ln x - x) - I + \int x\,dx$

Complete a second integration and substitute both limits correctly

Obtain correct answer $2\ln 2 - \frac{3}{4}$, or exact two-term equivalent **4**

81. M/J 02/P3/Q6

(i) State or imply $f(x) = \dfrac{A}{(3x+1)} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)}$

State or obtain $A = -3$

State or obtain $B = 2$

Use any relevant method to find C

Obtain $C = 1$ 5

[Special case: allow the form $\dfrac{A}{(3x+1)} + \dfrac{Dx+E}{(x+1)^2}$ and apply the above scheme ($A = -3$, $D = 1$, $E = 3$).]

[SR: if f(x) is given an incomplete form of partial fractions, give B1 or a form equivalent to the omission of C, or E, or B in the above, and M1 for finding one coefficient.]

(ii) Integrate and obtain terms $-\ln(3x + 1) - \dfrac{2}{(x+1)} + \ln(x + 1)$

Use limits correctly

Obtain the given answer correctly 5

82. M/J 02/P3/Q10

(i) State at any stage that the x-coordinate of A is equal to 1, or that A is the point (1, 0) 1

(ii) State $f'(x) = 2\dfrac{\ln x}{x}$, or equivalent

Use product or quotient rule for the next differentiation

Obtain $2 . \dfrac{1}{x} . \dfrac{1}{x} + 2\ln x . \left(\dfrac{-1}{x^2}\right)$, or any equivalent correct unsimplified form

Verify that $f''(e) = 0$ 4

(iii) State or imply area is $\int_1^e (\ln x)^2\,dx$

Use $\dfrac{dx}{du} = e^u$, or equivalent, in substituting for x throughout

Obtain given answer correctly (allow change of limits to be done mentally) 3

(iv) Attempt the first integration by parts, going the correct way

Obtain $(u^2 - 2u \pm 2)e^u$, or equivalent, after two applications of the rule

Obtain exact answer in terms of e, in any correct form, e.g. $(e - 2e + 2e) - 2$, or $e - 2$ 3

[The substitution in (iii) may be done in reverse i.e. starting with u integral and obtaining the x integral. The M1A1 scheme applies, but only an explicit statement will earn the B1.]

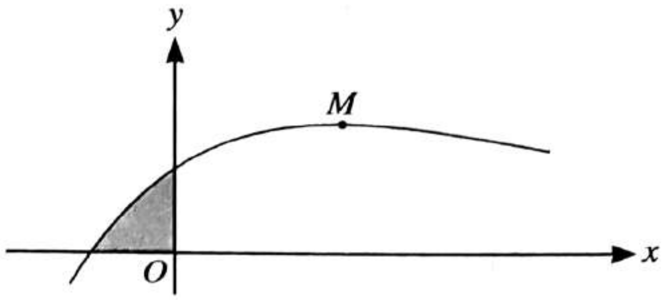
[The M1A1A1 in (iv) applies to those working in terms of x and obtaining $x((\ln x)^2 - 2\ln x \pm 2)$, or equivalent.]

5.2: Trapezium Rule

1. **M/J 18/P32/Q4/(i)**

   (i) Show that $\dfrac{2\sin x - \sin 2x}{1 - \cos 2x} \equiv \dfrac{\sin x}{1 + \cos x}$. **[4]**

2. **M/J 18/P32/Q8**



The diagram shows the curve $y = (x + 1)e^{-\frac{1}{3}x}$ and its maximum point M.

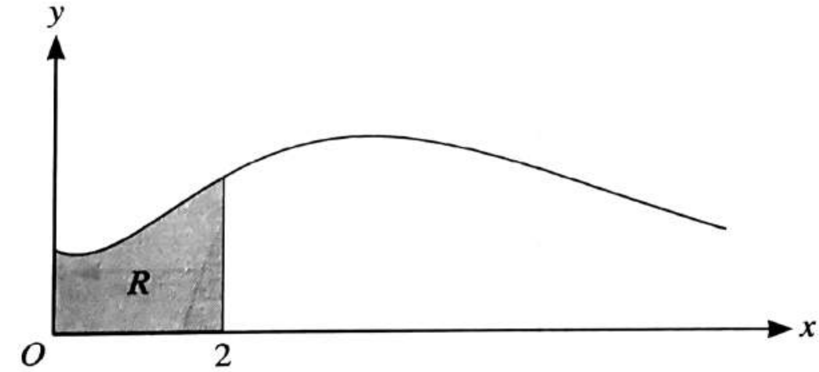
   (i) Find the x-coordinate of M. **[4]**

   (ii) Find the area of the shaded region enclosed by the curve and the axes, giving your answer in terms of e. **[5]**

3. **M/J 18/P33/Q7/(i)**

   (i) Express $\cos\theta + 2\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact values of R and $\tan\alpha$. **[3]**

4. **O/N 17/P31/Q9, O/N 17/P33/Q9**

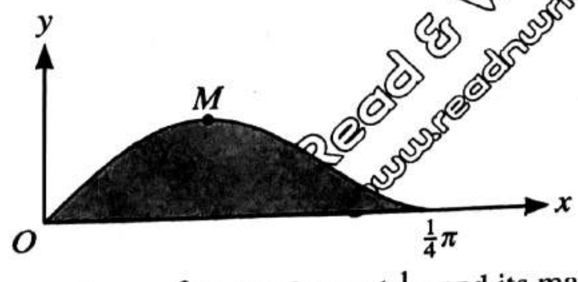


The diagram shows the curve $y = (1 + x^2)e^{-\frac{1}{2}x}$ for $x \geqslant 0$. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 2$.

   (i) Find the exact values of the x-coordinates of the stationary points of the curve. **[4]**

   (ii) Show that the exact value of the area of R is $18 - \dfrac{42}{e}$. **[5]**

5. **M/J 17/P31/Q10**

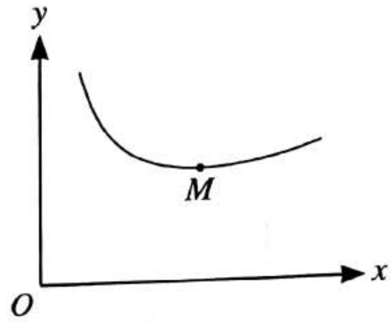


The diagram shows the curve $y = \sin x \cos^2 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$ and its maximum point M.

(i) Using the substitution $u = \cos x$, find by integration the exact area of the shaded region bounded by the curve and the x-axis. [6]

(ii) Find the x-coordinate of M. Give your answer correct to 2 decimal places. [6]

6. M/J 17/P33/Q7/(iii)



The diagram shows a sketch of the curve $y = \frac{e^{\frac{1}{2}x}}{x}$ for $x > 0$, and its minimum point M.

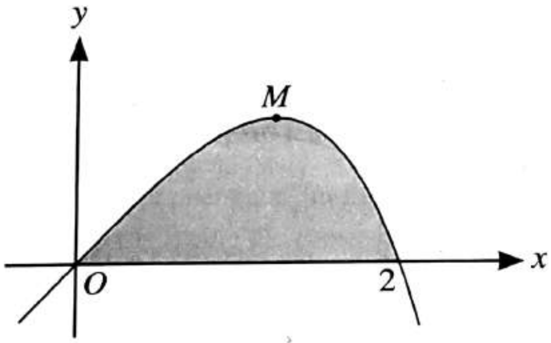
(i) The estimate found in part **(ii)** is denoted by E. Explain, without further calculation, whether another estimate found using the trapezium rule with four intervals would be greater than E or less than E. [1]

7. M/J 17/P33/Q9/(i)

Let $f(x) = \frac{3x^2 - 4}{x^2(3x+2)}$.

(i) Express $f(x)$ in partial fractions. [5]

8. O/N 16/P32/Q7, O/N16/P31/Q7



The diagram shows part of the curve $y = (2x - x^2)e^{\frac{1}{2}x}$ and its maximum point M.

(i) Find the exact x-coordinate of M. [4]

(ii) Find the exact value of the area of the shaded region bounded by the curve and the positive x-axis. [5]

9. M/J 15/P32/Q1

Use the trapezium rule with three intervals to estimate the value of

$$\int_0^{\frac{1}{2}\pi} \ln(1 + \sin x)\,dx,$$

giving your answer correct to 2 decimal places. [3]

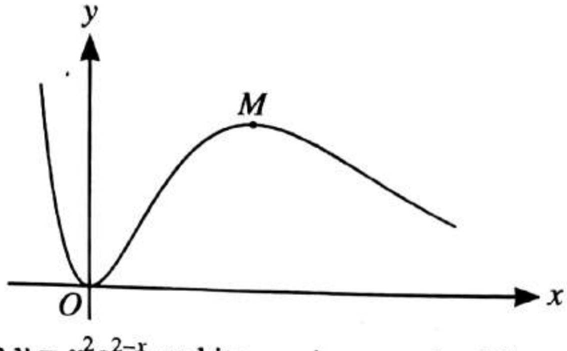
10. M/J 15/P31/Q2

Use the trapezium rule with three intervals to find an approximation to

$$\int_0^3 |3^x - 10|\,dx.$$

[4]

11. M/J 15/P31/Q9



The diagram shows the curve $y = x^2e^{2-x}$ and its maximum point M.

(i) Show that the x-coordinate of M is 2. **[3]**

(ii) Find the exact value of $\int_0^2 x^2e^{2-x}\,dx$. **[6]**

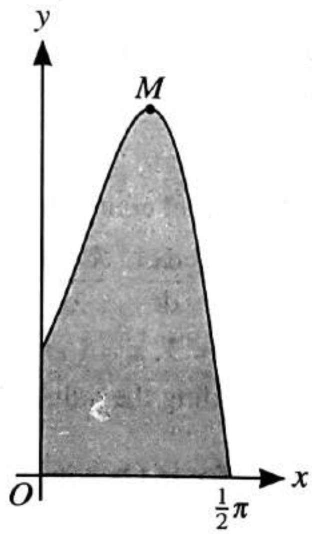
12. O/N 14/P33/Q6

It is given that $I = \int_0^{0.3} (1+3x^2)^{-2}\,dx$.

(i) Use the trapezium rule with 3 intervals to find an approximation to I, giving the answer correct to 3 decimal places. **[3]**

(ii) For small values of x, $(1+3x^2)^{-2} \approx 1 + ax^2 + bx^4$. Find the values of the constants a and b. Hence, by evaluating $\int_0^{0.3} (1 + ax^2 + bx^4)\,dx$, find a second approximation to I, giving the answer correct to 3 decimal places. **[5]**

13. M/J 14/P33/Q9

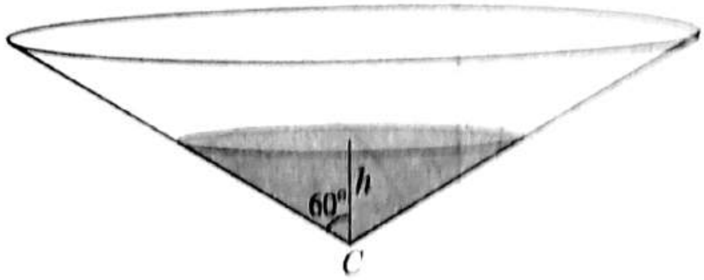


The diagram shows the curve $y = e^{2\sin x}\cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Using the substitution $u = \sin x$, find the exact value of the area of the shaded region bounded by the curve and the axes. **[5]**

(ii) Find the x-coordinate of M, giving your answer correct to 3 decimal places. **[6]**

14. O/N 13/P32/Q10



A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi-vertical angle is 60°, as shown in the diagram. At time $t = 0$, the tank is full and the depth of water is H. At this instant, a tap at C is opened and water begins to flow out. The volume of water in the tank decreases at a rate proportional to $\sqrt{h}$, where h is the depth of water at time t. The tank becomes empty when $t = 60$.

(i) Show that h and t satisfy a differential equation of the form

$$\frac{dh}{dt} = -Ah^{-\frac{3}{2}},$$

where A is a positive constant. [4]

(ii) Solve the differential equation given in part (i) and obtain an expression for t in terms of h and H. [6]

(iii) Find the time at which the depth reaches $\frac{1}{2}H$. [1]

[The volume V of a cone of vertical height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.]

15. M/J 13/P32/Q8/(ii)

(i) The variables x and y satisfy the differential equation [4]

$$y = x^2(2x+1)\frac{dy}{dx},$$

and $y = 1$ when $x = 1$. Solve the differential equation and find the exact value of y when $x = 2$. Give your value of y in a form not involving logarithms. [7]

16. M/J 13/P33/Q8

The variables x and t satisfy the differential equation

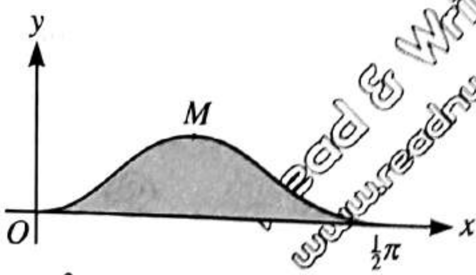
$$t\frac{dx}{dt} = \frac{k - x^3}{2x^2},$$

for $t > 0$, where k is a constant. When $t = 1$, $x = 1$ and when $t = 4$, $x = 2$.

(i) Solve the differential equation, finding the value of k and obtaining an expression for x in terms of t. [9]

(ii) State what happens to the value of x as t becomes large. [1]

17. M/J 13/P33/Q9



The diagram shows the curve $y = \sin^2 2x \cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Find the x-coordinate of M. [6]

(ii) Using the substitution $u = \sin x$, find by integration the area of the shaded region bounded by the curve and the x-axis. **[4]**

18. O/N 12/P32/Q6, O/N 12/P31/Q6

The variables x and y are related by the differential equation

$$x\frac{dy}{dx} = 1 - y^2.$$

When $x = 2$, $y = 0$. Solve the differential equation, obtaining an expression for y in terms of x. **[8]**

19. O/N 12/P33/Q4

The variables x and y are related by the differential equation

$$(x^2 + 4)\frac{dy}{dx} = 6xy.$$

It is given that $y = 32$ when $x = 0$. Find an expression for y in terms of x. **[6]**

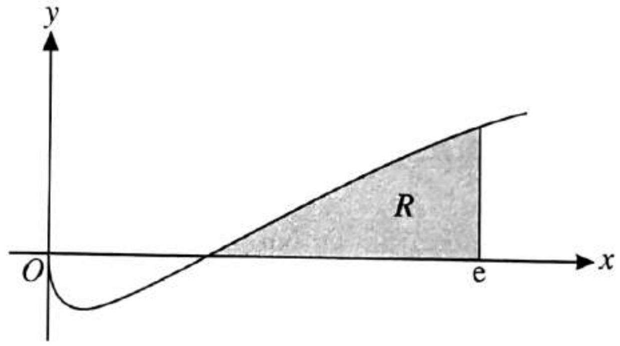
20. M/J 12/P32/Q5

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{2x+y},$$

and $y = 0$ when $x = 0$. Solve the differential equation, obtaining an expression for y in terms of x. **[6]**

21. M/J 12/P32/Q9

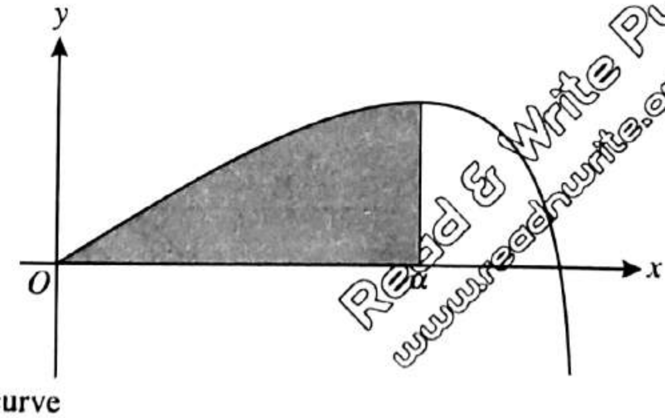


The diagram shows the curve $y = x^{\frac{1}{2}}\ln x$. The shaded region between the curve, the x-axis and the line $x = e$ is denoted by R.

(i) Find the equation of the tangent to the curve at the point where $x = 1$, giving your answer in the form $y = mx + c$. **[4]**

(ii) Find by integration the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π and e. **[7]**

22. M/J 12/P31/Q5



The diagram shows the curve

$$y = 8\sin\tfrac{1}{2}x - \tan\tfrac{1}{2}x$$

for $0 \leqslant x < \pi$. The x-coordinate of the maximum point is α and the shaded region is enclosed by the curve and the lines $x = \alpha$ and $y = 0$.

(i) Show that $\alpha = \frac{2}{3}\pi$. [3]

(ii) Find the exact value of the area of the shaded region. [4]

23. M/J 12/P31/Q7

The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \frac{6xe^{3x}}{y^2}.$$

It is given that $y = 2$ when $x = 0$. Solve the differential equation and hence find the value of y when $x = 0.5$, giving your answer correct to 2 decimal places. [8]

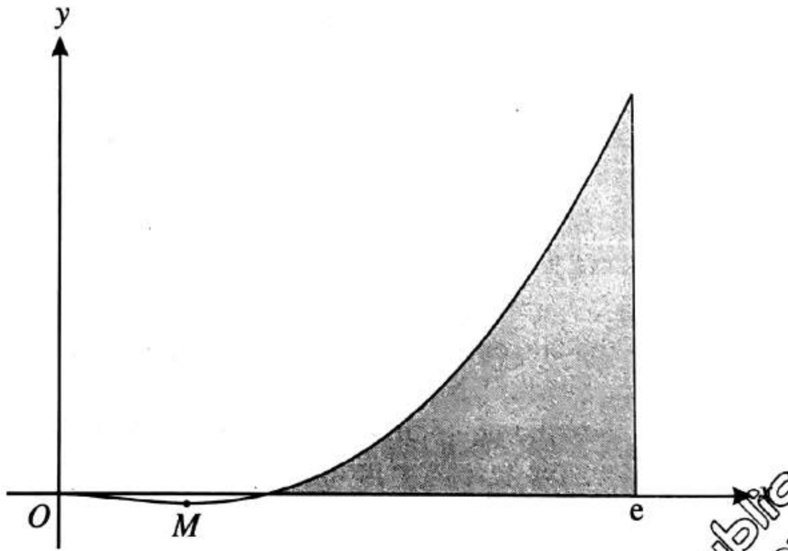
24. M/J 12/P33/Q5

In a certain chemical process a substance A reacts with another substance B. The masses in grams of A and B present at time t seconds after the start of the process are x and y respectively. It is given that $\frac{dy}{dt} = -0.6xy$ and $x = 5e^{-3t}$. When $t = 0$, $y = 70$.

(i) Form a differential equation in y and t. Solve this differential equation and obtain an expression for y in terms of t. [6]

(ii) The percentage of the initial mass of B remaining at time t is denoted by p. Find the exact value approached by p as t becomes large. [2]

25. O/N 11/P32/Q9, O/N 11/P31/Q9

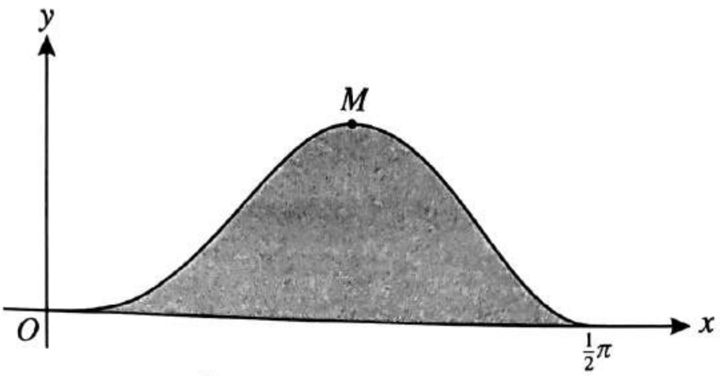


The diagram shows the curve $y = x^2 \ln x$ and its minimum point M.

(i) Find the exact values of the coordinates of M. [5]

(ii) Find the exact value of the area of the shaded region bounded by the curve, the x-axis and the line $x = e$. [5]

26. M/J 11/P33/Q8

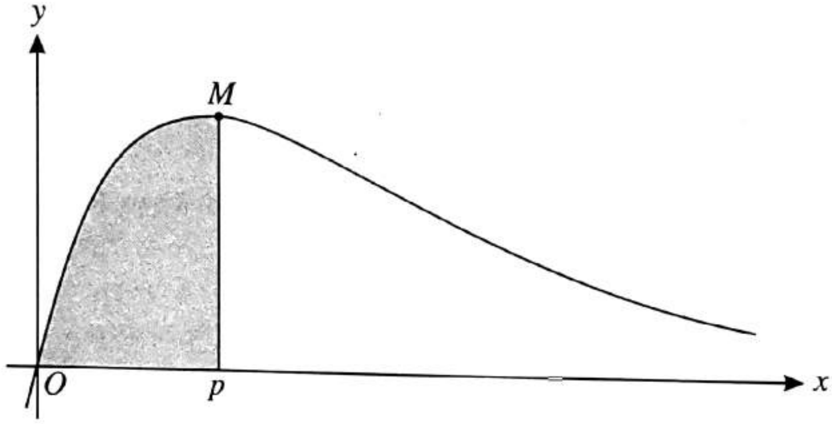


The diagram shows the curve $y = 5\sin^3 x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Find the x-coordinate of M. **[5]**

(ii) Using the substitution $u = \cos x$, find by integration the area of the shaded region bounded by the curve and the x-axis. **[5]**

27. M/J 10/P33/Q5

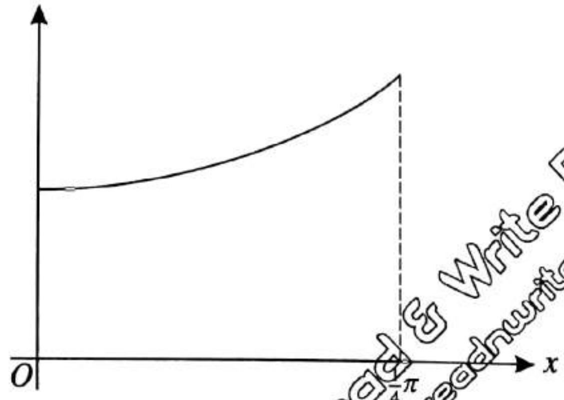


The diagram shows the curve $y = e^{-x} - e^{-2x}$ and its maximum point M. The x-coordinate of M is denoted by p.

(i) Find the exact value of p. **[4]**

(ii) Show that the area of the shaded region bounded by the curve, the x-axis and the line $x = p$ is equal to $\frac{1}{8}$. **[4]**

28. M/J 09/P3/Q2



The diagram shows the curve $y = \sqrt{(1 + 2\tan^2 x)}$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$.

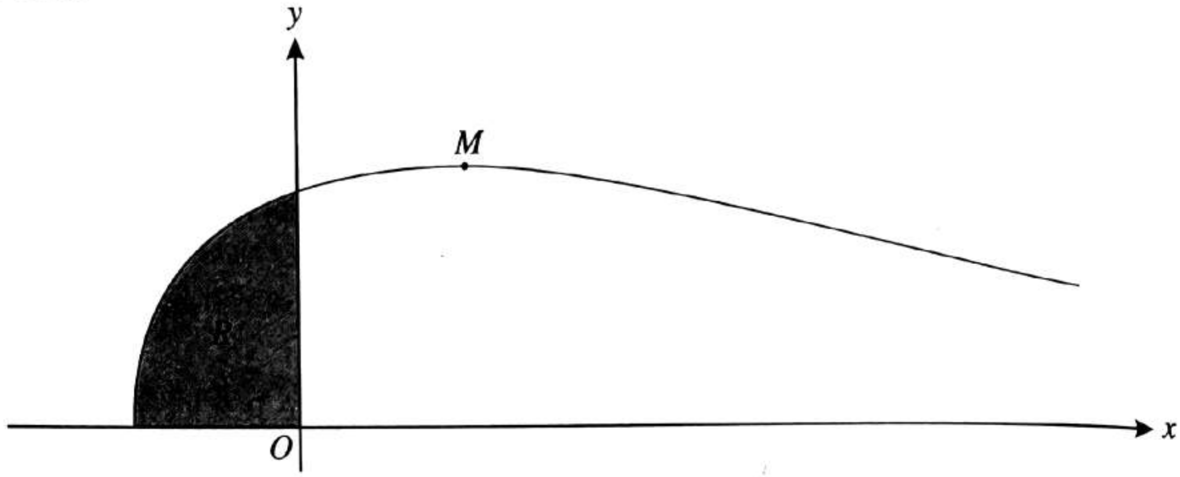
(i) Use the trapezium rule with three intervals to estimate the value of

$$\int_0^{\frac{1}{4}\pi} \sqrt{(1 + 2\tan^2 x)}\, dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) The estimate found in part (i) is denoted by E. Explain, without further calculation, whether another estimate found using the trapezium rule with six intervals would be greater than E or less than E. [1]

29. M/J 08/P3/Q9

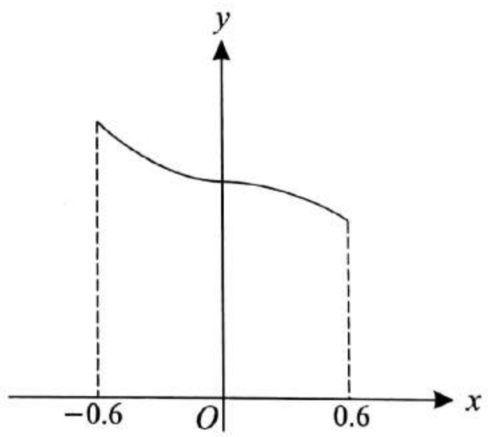


The diagram shows the curve $y = e^{-\frac{1}{2}x}\sqrt{(1+2x)}$ and its maximum point M. The shaded region between the curve and the axes is denoted by R.

(i) Find the x-coordinate of M. [4]

(ii) Find by integration the volume of the solid obtained when R is rotated completely about the x-axis. Give your answer in terms of π and e. [6]

30. M/J 05/P3/Q2



The diagram shows a sketch of the curve $y = \dfrac{1}{1+x^3}$ for values of x from -0.6 to 0.6.

(i) Use the trapezium rule, with two intervals, to estimate the value of

$$\int_{-0.6}^{0.6} \frac{1}{1+x^3}\,dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) Explain, with reference to the diagram, why the trapezium rule may be expected to give a good approximation to the true value of the integral in this case. [1]

Answers Section

1. M/J 18/P32/Q4/(i)

(i) Use correct double angle formulae and express LHS in terms of $\cos x$ and $\sin x$
Obtain a correct expression
Complete method to get correct denominator e.g. by factorising to remove a factor of $1-\cos x$
Obtain the given RHS correctly
OR (working R to L):

$$\frac{\sin x}{1+\cos x}\times\frac{1-\cos x}{1-\cos x}=\frac{\sin x-\sin x\cos x}{1-\cos^2 x}$$
$$=\frac{2\sin x-2\sin x\cos x}{2-2\cos^2 x}$$
$$=\frac{2\sin x-\sin 2x}{1-\cos 2x}$$ **4**

2. M/J 18/P32/Q8

(i) Use correct product or quotient rule
Obtain complete correct derivative in any form
Equate derivative to zero and solve for x
Obtain answer $x = 2$ with no errors seen **4**

(ii) Integrate by parts and reach $a(x+1)e^{-\frac{1}{3}x}+b\int e^{-\frac{1}{3}x}\,dx$

Obtain $-3(x+1)e^{-\frac{1}{3}x}+3\int e^{-\frac{1}{3}x}\,dx$, or equivalent

Complete integration and obtain $-3(x+1)e^{-\frac{1}{3}x}-9e^{-\frac{1}{3}x}$, or equivalent

Use **correct** limits $x = -1$ and $x = 0$ **in the correct order**, having integrated twice

Obtain answer $9e^{\frac{1}{3}}-12$, or equivalent **5**

3. M/J 18/P33/Q7/(i)

(i) State answer $R=\sqrt{5}$
Use trig formulae to find $\tan\alpha$
Obtain $\tan\alpha=2$ **3**

4. O/N 17/P31/Q9, O/N 17/P33/Q9

(i) Use correct product or quotient rule
Obtain correct derivative in any form
Equate derivative to zero and obtain a 3 term quadratic equation in x
Obtain answers $x=2\pm\sqrt{3}$ **4**

(ii) Integrate by parts and reach $k(1+x^2)e^{-\frac{1}{2}x}+l\int xe^{-\frac{1}{2}x}\,dx$

Obtain $-2(1+x^2)e^{-\frac{1}{2}x}+4\int xe^{-\frac{1}{2}x}\,dx$, or equivalent

Complete the integration and obtain $(-18-8x-2x^2)e^{-\frac{1}{2}x}$, or equivalent
Use limits $x=0$ and $x=2$ correctly, having fully integrated twice by parts
Obtain the given answer 5

5. M/J 17/P31/Q10

(i) State or imply $du = -\sin x \, dx$
Using correct double angle formula, express the integral in terms of u and du
Obtain integrand $\pm(2u^2-1)^2$
Change limits and obtain correct integral $\int_{\frac{1}{\sqrt{2}}}^{1}(2u^2-1)^2 du$ with no errors seen
Substitute limits in an integral of the form au^5+bu^3+cu
Obtain answer $\frac{1}{15}(7-4\sqrt{2})$, or exact simplified equivalent 6

(ii) Use product rule and chain rule at least once
Obtain correct derivative in any form
Equate derivative to zero and use trig formulae to obtain an equation in $\cos x$ and $\sin x$
Use correct methods to obtain an equation in $\cos x$ or $\sin x$ only
Obtain $10\cos^2 x = 9$ or $10\sin^2 x = 1$, or equivalent
Obtain answer 0.32 6

6. M/J 17/P33/Q7/(iii)

(i) Explain why the estimate would be less than E 1

7. M/J 17/P33/Q9/(i)

(i) State or imply the form $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{3x+2}$
Use a relevant method to determine a constant
Obtain one of the values $A=3$, $B=-2$, $C=-6$
Obtain a second value
Obtain the third value
[Mark the form $\frac{Ax+B}{x^2}+\frac{C}{3x+2}$ using same pattern of marks.] 5

8. O/N 16/P32/Q7, O/N16/P31/Q7

(i) Use the correct product rule
Obtain correct derivative in any form, e.g. $(2-2x)e^{\frac{1}{2}x}+\frac{1}{2}(2x-x^2)e^{\frac{1}{2}x}$
Equate derivative to zero and solve for x
Obtain $x=\sqrt{5}-1$ only [4]

(ii) Integrate by parts and reach $a(2x-x^2)e^{\frac{1}{2}x}+b\int(2-2x)e^{\frac{1}{2}x}dx$
Obtain $2e^{\frac{1}{2}x}(2x-x^2)-2\int(2-2x)e^{\frac{1}{2}x}dx$, or equivalent
Complete the integration correctly, obtaining $(12x-2x^2-24)e^{\frac{1}{2}x}$, or equivalent
Use limits $x=0$, $x=2$ correctly having integrated by parts twice
Obtain answer $24-8e$, or exact simplified equivalent [5]

9. M/J 15/P32/Q1

State or imply ordinates 0, 0.405465…, 0.623810…, 0.693147…

Use correct formula, or equivalent, with $h = \frac{1}{6}\pi$ and four ordinates

Obtain answer 0.72 **[3]**

10. M/J 15/P31/Q2

Attempt calculation of at least 3 ordinates

Obtain 9, 7, 1, 17

Use trapezium rule with $h = 1$

Obtain $\frac{1}{2}(9+14+2+17)$ or equivalent and hence 21 **[4]**

11. M/J 15/P31/Q9

(i) Use product rule to find first derivative

Obtain $2xe^{2-x} - x^2e^{2-x}$

Confirm $x = 2$ at M **[3]**

(ii) Attempt integration by parts and reach $\pm x^2e^{2-x} \pm \int 2xe^{2-x}\,dx$

Obtain $-x^2e^{2-x} + \int 2xe^{2-x}\,dx$

Attempt integration by parts and reach $\pm x^2e^{2-x} \pm 2xe^{2-x} \pm 2e^{2-x}$

Obtain $-x^2e^{2-x} - 2xe^{2-x} - 2e^{2-x}$

Use limits 0 and 2 having integrated twice

Obtain $2e^2 - 10$ **[6]**

12. O/N 14/P33/Q6

(i) State or imply correct ordinates 1, 0.94259…, 0.79719…, 0.62000…

Use correct formula or equivalent with $h = 0.1$ and four y values

Obtain 0.255 with no errors seen **[3]**

(ii) Obtain or imply $a = -6$

Obtain x^4 term including correct attempt at coefficient

Obtain or imply $b = 27$

Either Integrate to obtain $x - 2x^3 + \frac{27}{5}x^5$, following their values of a and b

Obtain 0.259

Or Use correct trapezium rule with at least 3 ordinates

Obtain 0.259 (from 4) **[5]**

13. M/J 14/P33/Q9

(i) Substitute for x and dx throughout using $u = \sin x$ and $du = \cos x\,dx$, or equivalent

Obtain integrand e^{2u}

Obtain indefinite integral $\frac{1}{2}e^{2u}$

Use limits $u = 0$, $u = 1$ correctly, or equivalent

Obtain answer $\frac{1}{2}(e^2 - 1)$, or exact equivalent **5**

(ii) Use chain rule or product rule

Obtain correct terms of the derivative in any form, e.g. $2\cos x\, e^{2\sin x}\cos x - e^{2\sin x}\sin x$

Equate derivative to zero and obtain a quadratic equation in $\sin x$

Solve a 3-term quadratic and obtain a value of x

Obtain answer 0.896 **6**

14. O/N 13/P32/Q10

(i) State or imply $V = \pi h^3$

State or imply $\frac{dV}{dt} = -k\sqrt{h}$

Use $\frac{dV}{dt} = \frac{dV}{dh}.\frac{dh}{dt}$, or equivalent

Obtain the given equation

[The M1 is only available if $\frac{dV}{dh}$ is in terms of h and has been obtained by a correct method.] [4]

[Allow B1 for $\frac{dV}{dt} = k\sqrt{h}$ but withhold the final A1 until the polarity of the constant $\frac{k}{3\pi}$ has been justified.]

(ii) Separate variables and integrate at least one side

Obtain terms $\frac{2}{5}h^{\frac{5}{2}}$ and $-At$, or equivalent

Use $t = 0, h = H$ in a solution containing terms of the form $ah^{\frac{5}{2}}$ and $bt + c$

Use $t = 60, h = 0$ in a solution containing terms of the form $ah^{\frac{5}{2}}$ and $bt + c$

Obtain a correct solution in any form, e.g. $\frac{2}{5}h^{\frac{5}{2}} = \frac{1}{150}H^{\frac{5}{2}}t + \frac{2}{5}H^{\frac{5}{2}}$

(ii) Obtain final answer $t = 60\left(1 - \left(\frac{h}{H}\right)^{\frac{5}{2}}\right)$, or equivalent [6]

(iii) Substitute $h = \frac{1}{2}H$ and obtain answer $t = 49.4$ [1]

15. M/J 13/P32/Q8/(ii)

(i) Separate variables and obtain one term by integrating $\frac{1}{y}$ or a partial fraction

Obtain $\ln y = -\frac{1}{2} - 2\ln(2x + 1) + c$, or equivalent

Evaluate a constant, or use limits $x = 1, y = 1$, in a solution containing at least three terms of the form $k \ln y$, l/x, $m \ln x$ and $n \ln(2x + 1)$, or equivalent

Obtain solution $\ln y = -\frac{1}{2} - 2\ln x + 2\ln(2x+1) + c$, or equivalent

Substitute $x = 2$ and obtain $y = \frac{25}{36}e^{\frac{1}{2}}$, or exact equivalent free of logarithms [7]

(The f.t. is on A, B, C. Give A2✓ if there is only one error or omission in the integration; A1✓ if two.)

16. M/J 13/P33/Q8

(i) Separate variables correctly and integrate at least one side

Obtain term $\ln t$, or equivalent

Obtain term of the form $a\ln(k - x^3)$

Obtain term $-\frac{2}{3}\ln(k-x^3)$, or equivalent

EITHER: Evaluate a constant or use limits $t = 1, x = 1$ in a solution containing $a \ln t$ and $b \ln(k-x^3)$

Obtain correct answer in any form e.g. $\ln t = -\frac{2}{3}\ln(k-x^3)+\frac{2}{3}\ln(k-1)$

Use limits $t = 4, x = 2$, and solve for k
Obtain $k = 9$

OR: Using limits $t = 1, x = 1$ and $t = 4, x = 2$ in a solution containing $a \ln t$ and $b \ln (k-x^3)$ obtain an equation in k

Obtain a correct equation in any form, e.g. $\ln 4 = -\frac{2}{3}\ln(k-8)+\frac{2}{3}\ln(k-1)$

Solve for k
Obtain $k = 9$

Substitute $k = 9$ and obtain $x = (9-8t^{-\frac{3}{2}})^{\frac{1}{3}}$ **[9]**

(ii) State that x approaches $9^{\frac{1}{3}}$, or equivalent **[1]**

17. M/J 13/P33/Q9

(i) Use product rule
Obtain correct derivative in any form, e.g. $4\sin 2x \cos 2x \cos x - \sin^2 2x \sin x$
Equate derivative to zero and use a double angle formula
Reduce equation to one in a single trig function
Obtain a correct equation in any form,
e.g. $10 \cos^3 x = 6 \cos x$, $4 = 6 \tan^2 x$ or $4 = 10 \sin^2 x$
Solve and obtain $x = 0.685$ **[6]**

(ii) Using $du = \pm \cos x \, dx$, or equivalent, express integral in terms of u and du
Obtain $\int 4u^2(1-u^2)\,du$, or equivalent
Use limits $u = 0$ and $u = 1$ in an integral of the form $au^3 + bu^5$
Obtain answer $\frac{8}{15}$ (or 0.533) **[4]**

18. O/N 12/P32/Q6, O/N 12/P31/Q6

Separate variables correctly and attempt integration of one side
Obtain term $\ln x$
State or imply $\frac{1}{1-y^2} \equiv \frac{A}{1-y} + \frac{B}{1+y}$ and use a relevant method to find A or B
Obtain $A = \frac{1}{2}, B = \frac{1}{2}$
Integrate and obtain $-\frac{1}{2}\ln(1-y) + \frac{1}{2}\ln(1+y)$, or equivalent
[If the integral is directly stated as $k_1 \ln\left(\frac{1+y}{1-y}\right)$ or $k_2 \ln\left(\frac{1-y}{1+y}\right)$ give M1, and then A2 for $k_1 = \frac{1}{2}$ or $k_2 = -\frac{1}{2}$]
Evaluate a constant, or use limits $x = 2, y = 0$ in a solution containing terms $a \ln x$, $b \ln(1-y)$ and $c \ln(1+y)$, where $abc \neq 0$
[This M mark is not available if the integral of $1/(1-y^2)$ is initially taken to be of the form $k \ln(1-y^2)$]
Obtain solution in any correct form, e.g. $\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) = \ln x - \ln 2$
Rearrange and obtain $y = \frac{x^2-4}{x^2+4}$, or equivalent, free of logarithms **[8]**

19. O/N 12/P33/Q4

Separate variables correctly and integrate one side

Obtain $\ln y = \ldots$ or equivalent

Obtain $= 3\ln(x^2+4)$ or equivalent

Evaluate a constant or use $x = 0, y = 32$ as limits in a solution containing terms $a \ln y$ and $b\ln\left(x^2+4\right)$

Obtain $\ln y = 3\ln(x^2+4) + \ln 32 - 3\ln 4$ or equivalent

Obtain $y = \frac{1}{2}\left(x^2+4\right)$ or equivalent [6]

20. M/J 12/P32/Q5

Separate variables correctly and attempt integration of both sides

Obtain term $-e^{-y}$, or equivalent

Obtain term $\frac{1}{2}e^{2x}$, or equivalent

Evaluate a constant, or use limits $x = 0, y = 0$ in a solution containing terms ae^{-y} and be^{2x}

Obtain correct solution in any form, e.g. $-e^{-y} = \frac{1}{2}e^{2x} - \frac{3}{2}$

Rearrange and obtain $y = \ln(2/(3 - e^{2x}))$, or equivalent [6]

21. M/J 12/P32/Q9

(i) Use correct product rule

Obtain derivative in any correct form, e.g. $\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}$

Carry out a complete method to form an equation of the tangent at $x = 1$

Obtain answer $y = x - 1$ [4]

(ii) State or imply that the indefinite integral for the volume is $\pi\int x(\ln x)^2\,dx$

Integrate by parts and reach $ax^2(\ln x)^2 + b\int x^2 . \frac{\ln x}{x}\,dx$

Obtain $\frac{1}{2}x^2(\ln x)^2 - \int x\ln x\,dx$, or unsimplified equivalent

Attempt second integration by parts reaching $cx^2 \ln x + d\int x^2 . \frac{1}{x}\,dx$

Complete the integration correctly, obtaining $\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2$

Substitute limits $x = 1$ and $x = e$, having integrated twice

Obtain answer $\frac{1}{4}\pi(e^2 - 1)$, or exact equivalent [7]

[If π omitted, or 2π or π/2 used, give B0 and then follow through.]
[Integration using parts $x \ln x$ and $\ln x$ is also viable.]

22. M/J 12/P31/Q5

(i) Differentiate to obtain $4\cos\frac{1}{2}x - \frac{1}{2}\sec^2\frac{1}{2}x$

Equate to zero and find value of $\cos\frac{1}{2}x$

Obtain $\cos\frac{1}{2}x=\frac{1}{2}$ and confirm $\alpha=\frac{2}{3}\pi$ [3]

(ii) Integrate to obtain $-16\cos\frac{1}{2}x\ldots$

$\ldots+2\ln\cos\frac{1}{2}x$ or equivalent

Using limits 0 and $\frac{2}{3}\pi$ in $a\cos\frac{1}{2}x+b\ln\cos\frac{1}{2}x$

Obtain $8+2\ln\frac{1}{2}$ or exact equivalent [4]

23. M/J 12/P31/Q7

Separate variables correctly and attempt integration on at least one side

Obtain $\frac{1}{3}y^3$ or equivalent on left-hand side

Use integration by parts on right-hand side (as far as $axe^{3x}+\int be^{3x}\,dx$)

Obtain or imply $2xe^{3x}+\int 2e^{3x}\,dx$ or equivalent

Obtain $2xe^{3x}-\frac{2}{3}e^{3x}$

Substitute $x=0, y=2$ in an expression containing terms Ay^3, Bxe^{3x}, Ce^{3x}, where $ABC\neq 0$, and find the value of c

Obtain $\frac{1}{3}y^3=2xe^{3x}-\frac{2}{3}e^{3x}+\frac{10}{3}$ or equivalent

Substitute $x=0.5$ to obtain $y=2.44$ [8]

24. M/J 12/P33/Q5

(i) Substitute for x, separate variables correctly and attempt integration of both sides
Obtain term $\ln y$, or equivalent
Obtain term e^{-3t}, or equivalent
Evaluate a constant, or use $t=0$, $y=70$ as limits in a solution containing terms $a\ln y$ and be^{-3t}
Obtain correct solution in any form, e.g. $\ln y-\ln 70=e^{-3t}-1$
Rearrange and obtain $y=70\exp(e^{-3t}-1)$, or equivalent [6]

(ii) Using answer to part (i), either express p in terms of t or use $e^{-3t}\to 0$ to find the limiting value of y
Obtain answer $\frac{100}{e}$ from correct exact work [2]

25. O/N 11/P32/Q9, O/N 11/P31/Q9

(i) Use product rule
Obtain correct derivative in any form
Equate derivative to zero and solve for x
Obtain answer $x=e^{-\frac{1}{2}}$, or equivalent
Obtain answer $y=-\frac{1}{2}e^{-1}$, or equivalent [5]

(ii) Attempt integration by parts reaching $kx^3\ln x\pm k\int x^3.\frac{1}{x}dx$

Obtain $\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx$, or equivalent
Integrate again and obtain $\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3$, or equivalent
Use limits $x = 1$ and $x = e$, having integrated twice
Obtain answer $\frac{1}{9}(2e^3 + 1)$, or exact equivalent [5]
[SR: An attempt reaching $ax^2(x \ln x - x) + b\int 2x(x \ln x - x)\, dx$ scores M1. Then give the first A1 for $I = x^2(x \ln x - x) - 2I + \int 2x^2 \, dx$, or equivalent.]

26. M/J 11/P33/Q8

(i) Use product and chain rule
Obtain correct derivative in any form, e.g. $15\sin^2 x\cos^3 x - 10\sin^4 x\cos x$
Equate derivative to zero and obtain a relevant equation in one trigonometric function
Obtain $2\tan^2 x = 3$, $5\cos^2 x = 2$, or $5\sin^2 x = 3$
Obtain answer $x = 0.886$ radians [5]

(ii) State or imply $du = -\sin x \, dx$, or $\frac{du}{dx} = -\sin x$, or equivalent
Express integral in terms of u and du
Obtain $\pm\int 5(u^2 - u^4)\, du$, or equivalent
Integrate and use limits $u = 1$ and $u = 0$ (or $x = 0$ and $x = \frac{1}{2}\pi$)
Obtain answer $\frac{2}{3}$, or equivalent, with no errors seen [5]

27. M/J 10/P33/Q5

(i) State derivative $-e^{-x} - (-2)e^{-2x}$, or equivalent
Equate derivative to zero and solve for x
Obtain $p = \ln 2$, or exact equivalent [4]

(ii) State indefinite integral $-e^{-x} - (-\frac{1}{2})e^{-2x}$, or equivalent
Substitute limits $x = 0$ and $x = p$ correctly
Obtain given answer following full and correct working [4]

28. M/J 09/P3/Q2

(i) State or imply 3 of the 4 ordinates 1, 1.069389..., 1.290994..., 1.732050...
Use correct formula, or equivalent, with $h = \frac{1}{12}\pi$ and four ordinates
Obtain answer 0.98 with no errors seen 3
[Accept $h = 0.26$ but not $h = 15$ when awarding the M1]
[SR: if only $\sqrt{\frac{5}{3}}$ and/or $\sqrt{3}$ are given, and decimals are not seen, the B1 is available]
[SR: solutions with 2 or 4 intervals can score only the M1 for a correct expression]

(ii) Justify statement that the second estimate would be less than E 1

29. M/J 08/P3/Q9

(i) Either use correct product or quotient rule, or square both sides, use correct product rule and make a reasonable attempt at applying the chain rule
Obtain correct result of differentiation in any form
Set derivative equal to zero and solve for x

Obtain $x=\frac{1}{2}$ only, correctly **[4]**

(ii) State or imply the indefinite integral for the volume is $\pi\int e^{-x}(1+2x)dx$

Integrate by parts and reach $\pm e^{-x}(1+2x)\pm\int 2e^{-x}dx$

Obtain $-e^{-x}(1+2x)+\int 2e^{-x}dx$, or equivalent

Complete integration correctly, obtaining $-e^{-x}(1+2x)-2e^{-x}$, or equivalent

Use limits $x=-\frac{1}{2}$ and $x=0$ correctly, having integrated twice

Obtain exact answer $\pi(2\sqrt{e}-3)$, or equivalent **[6]**

[If π omitted initially or 2π or $\pi/2$ used, give B0 and then follow through.]

30. M/J 05/P3/Q2

(i) Show or imply correct decimal ordinates 1.2755…, 1, 0.8223…
Use correct formula, or equivalent, with h = 0.6 and three ordinates
Obtain correct answer 1.23 with no errors seen 3
[SR: if the area is calculated with one interval, or three or more, give for a correct answer.]

(ii) Give an adequate justification, e.g. one trapezium over-estimates area and the other under-estimates, or errors cancel out 1

UNIT 6

Numerical Solution of Equations

A-Level
Mathematics Paper 3
Topical Workbook

READ & WRITE PUBLICATIONS

Sale Point: Shop # 25-28 Lower Ground Floor,
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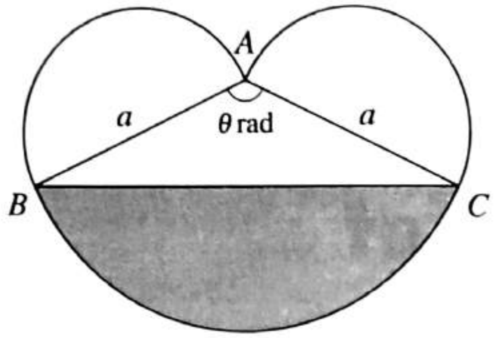
Unit-6: Numerical Solution of Equations

1. M/J 18/P32/Q6



The diagram shows a triangle ABC in which $AB = AC = a$ and angle $BAC = \theta$ radians. Semicircles are drawn outside the triangle with AB and AC as diameters. A circular arc with centre A joins B and C. The area of the shaded segment is equal to the sum of the areas of the semicircles.

(i) Show that $\theta = \frac{1}{2}\pi + \sin\theta$. [3]

(ii) Verify by calculation that θ lies between 2.2 and 2.4. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to determine θ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

2. M/J 18/P31/Q8

The positive constant a is such that $\int_0^a x e^{-\frac{1}{2}x}\, dx = 2$.

(i) Show that a satisfies the equation $a = 2\ln(a + 2)$. [5]

(ii) Verify by calculation that a lies between 3 and 3.5. [2]

(iii) Use an iteration based on the equation in part **(i)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

3. M/J 18/P33/Q4

The curve with equation $y = \frac{\ln x}{3 + x}$ has a stationary point at $x = p$.

(i) Show that p satisfies the equation $\ln x = 1 + \frac{3}{x}$. [3]

(ii) By sketching suitable graphs, show that the equation in part **(i)** has only one root. [2]

(iii) It is given that the equation in part **(i)** can be written in the form $x = \frac{3 + x}{\ln x}$. Use an iterative formula based on this rearrangement to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

4. O/N 17/P31/Q3, O/N 17/P33/Q3

The equation $x^3 = 3x + 7$ has one real root, denoted by α.

(i) Show by calculation that α lies between 2 and 3. [2]

Two iterative formulae, A and B, derived from this equation are as follows:

$$x_{n+1} = (3x_n + 7)^{\frac{1}{3}}, \qquad (A)$$

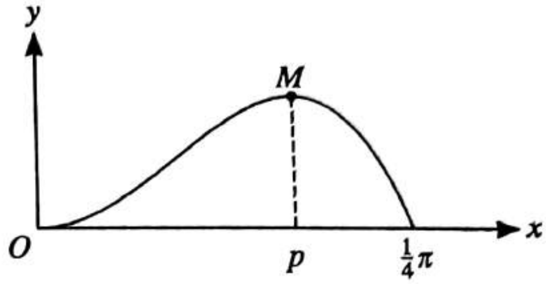
$$x_{n+1} = \frac{x_n^3 - 7}{3}. \qquad (B)$$

Each formula is used with initial value $x_1 = 2.5$.

(ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [4]

5. M/J 17/P32/Q10

The diagram shows the curve $y = x^2 \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The curve has a maximum point at M where $x = p$.

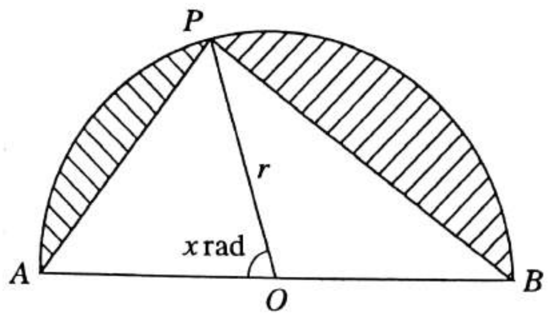


(i) Show that p satisfies the equation $p = \frac{1}{2}\tan^{-1}\left(\frac{1}{p}\right)$. [3]

(ii) Use the iterative formula $p_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{1}{p_n}\right)$ to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) Find, showing all necessary working, the exact area of the region bounded by the curve and the x-axis. [5]

6. M/J 17/P31/Q5



The diagram shows a semicircle with centre O, radius r and diameter AB. The point P on its circumference is such that the area of the minor segment on AP is equal to half the area of the minor segment on BP. The angle AOP is x radians.

(i) Show that x satisfies the equation $x = \frac{1}{3}(\pi + \sin x)$. [3]

(ii) Verify by calculation that x lies between 1 and 1.5. [2]

(iii) Use an iterative formula based on the equation in part (i) to determine x correct to 3 decimal places. Give the result of each iteration to 5 decimal places . [3]

7. M/J 17/P33/Q6(i, ii)

The equation $\cot x = 1 - x$ has one root in the interval $0 < x < \pi$, denoted by α.

(i) Show by calculation that α is greater than 2.5. [2]

(ii) Show that, if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula $x_{n+1} = \pi + \tan^{-1}\left(\frac{1}{1 \quad x}\right)$ converges, then it converges to α. [2]

8. O/N 16/P32/Q6, O/N 16/P31/Q6

(i) By sketching a suitable pair of graphs, show that the equation

$$\operatorname{cosec} \tfrac{1}{2}x = \tfrac{1}{3}x + 1$$

has one root in the interval $0 < x \leqslant \pi$. [2]

(ii) Show by calculation that this root lies between 1.4 and 1.6. [2]

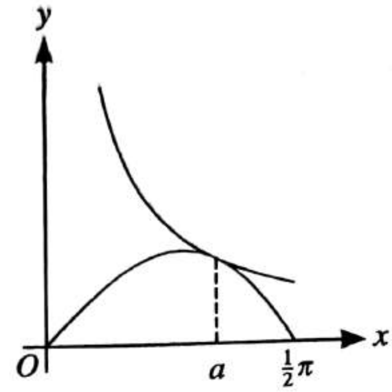
(iii) Show that, if a sequence of values in the interval $0 < x \leqslant \pi$ given by the iterative formula

$$x_{n+1} = 2\sin^{-1}\left(\frac{3}{x_n + 3}\right)$$

converges, then it converges to the root of the equation in part (i). [2]

(iv) Use this iterative formula to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

9. O/N 16/P33/Q9



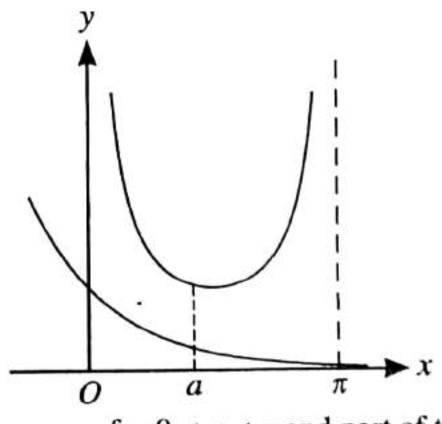
The diagram shows the curves $y = x\cos x$ and $y = \frac{k}{x}$, where k is a constant, for $0 < x \leqslant \frac{1}{2}\pi$. The curves touch at the point where $x = a$.

(i) Show that a satisfies the equation $\tan a = \frac{2}{a}$. [5]

(ii) Use the iterative formula $a_{n+1} = \tan^{-1}\left(\frac{2}{a_n}\right)$ to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iii) Hence find the value of k correct to 2 decimal places. [2]

10. O/N 16/P32/Q8



The diagram shows the curve $y = \operatorname{cosec} x$ for $0 < x < \pi$ and part of the curve $y = e^{-x}$. When $x = a$, the tangents to the curves are parallel.

(i) By differentiating $\frac{1}{\sin x}$, show that if $y = \operatorname{cosec} x$ then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$. [3]

(ii) By equating the gradients of the curves at $x = a$, show that

$$a = \tan^{-1}\left(\frac{e^a}{\sin a}\right).$$ [2]

(iii) Verify by calculation that a lies between 1 and 1.5. [2]

(iv) Use an iterative formula based on the equation in part (ii) to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

11. M/J 16/P33/Q6

The curve with equation $y = x^2 \cos \frac{1}{2}x$ has a stationary point at $x = p$ in the interval $0 < x < \pi$.

(i) Show that p satisfies the equation $\tan \frac{1}{2}p = \frac{4}{p}$. [3]

(ii) Verify by calculation that p lies between 2 and 2.5. [2]

(iii) Use the iterative formula $p_{n+1} = 2\tan^{-1}\left(\frac{4}{p_n}\right)$ to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

12. O/N 15/P32/Q4

The equation $x^3 - x^2 - 6 = 0$ has one real root, denoted by α.

(i) Find by calculation the pair of consecutive integers between which α lies. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{\left(x_n + \frac{6}{x_n}\right)}$$

converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

13. O/N 15/P33/Q4

A curve has parametric equations

$$x = t^2 + 3t + 1, \qquad y = t^4 + 1.$$

The point P on the curve has parameter p. It is given that the gradient of the curve at P is 4.

(i) Show that $p = \sqrt[3]{(2p + 3)}$. [3]

(ii) Verify by calculation that the value of p lies between 1.8 and 2.0. [2]

(iii) Use an iterative formula based on the equation in part (i) to find the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

14. O/N 15/P31/Q4

The equation $x^3 - x^2 - 6 = 0$ has one real root, denoted by α.

(i) Find by calculation the pair of consecutive integers between which α lies. [2]

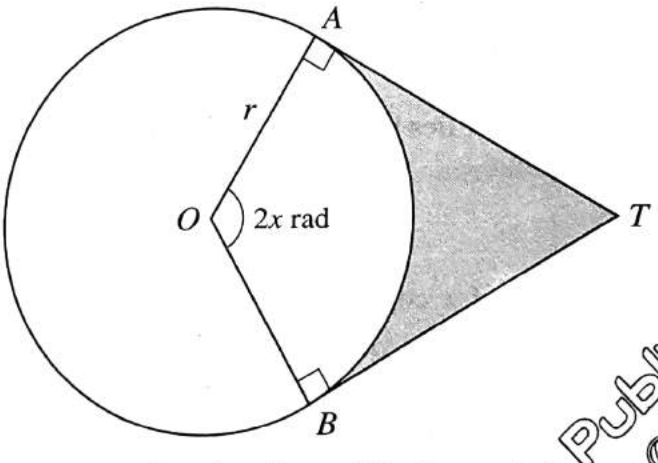
(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{\left(x_n + \frac{6}{x_n}\right)}$$

converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

15. M/J 15/P32/Q5



The diagram shows a circle with centre O and radius r. The tangents to the circle at the points A and B meet at T, and the angle AOB is $2x$ radians. The shaded region is bounded by the tangents AT and BT, and by the minor arc AB. The perimeter of the shaded region is equal to the circumference of the circle.

(i) Show that x satisfies the equation

$$\tan x = \pi - x.$$ [3]

(ii) This equation has one root in the interval $0 < x < \frac{1}{2}\pi$. Verify by calculation that this root lies between 1 and 1.3. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi - x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

16. O/N 14/P33/Q9

(i) Sketch the curve $y = \ln(x+1)$ and hence, by sketching a second curve, show that the equation

$$x^3 + \ln(x+1) = 40$$

has exactly one real root. State the equation of the second curve. [3]

(ii) Verify by calculation that the root lies between 3 and 4. [2]

(iii) Use the iterative formula

$$x_{n+1} = \sqrt[3]{(40 - \ln(x_n + 1))},$$

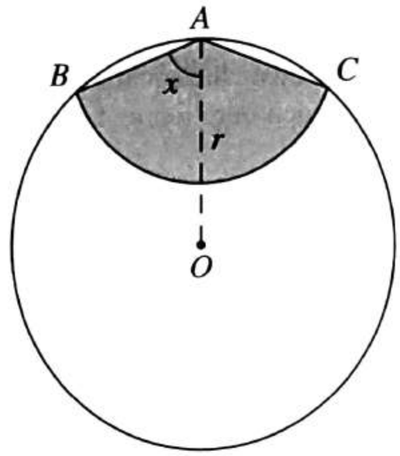
with a suitable starting value, to find the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iv) Deduce the root of the equation

$$(e^y - 1)^3 + y = 40,$$

giving the answer correct to 2 decimal places. [2]

17. M/J 14/P32/Q6



In the diagram, A is a point on the circumference of a circle with centre O and radius r. A circular arc with centre A meets the circumference at B and C. The angle OAB is equal to x radians. The shaded region is bounded by AB, AC and the circular arc with centre A joining B and C. The perimeter of the shaded region is equal to half the circumference of the circle.

(i) Show that $x = \cos^{-1}\left(\dfrac{\pi}{4 + 4x}\right)$. [3]

(ii) Verify by calculation that x lies between 1 and 1.5. [2]

(iii) Use the iterative formula

$$x_{n+1} = \cos^{-1}\left(\frac{\pi}{4 + 4x_n}\right)$$

to determine the value of x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

18. M/J 14/P32/Q6

(i) By sketching each of the graphs $y = \operatorname{cosec} x$ and $y = x(\pi - x)$ for $0 < x < \pi$, show that the equation

$$\operatorname{cosec} x = x(\pi - x)$$

has exactly two real roots in the interval $0 < x < \pi$. [3]

(ii) Show that the equation $\operatorname{cosec} x = x(\pi - x)$ can be written in the form $x = \dfrac{1 + x^2 \sin x}{\pi \sin x}$. [2]

(iii) The two real roots of the equation $\operatorname{cosec} x = x(\pi - x)$ in the interval $0 < x < \pi$ are denoted by α and β, where $\alpha < \beta$.

(a) Use the iterative formula

$$x_{n+1} = \frac{1 + x_n^2 \sin x_n}{\pi \sin x_n}$$

to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(b) Deduce the value of β correct to 2 decimal places. [1]

19. M/J 14/P33/Q4

The equation $x = \dfrac{10}{e^{2x} - 1}$ has one positive real root, denoted by α.

(i) Show that α lies between $x = 1$ and $x = 2$. [2]

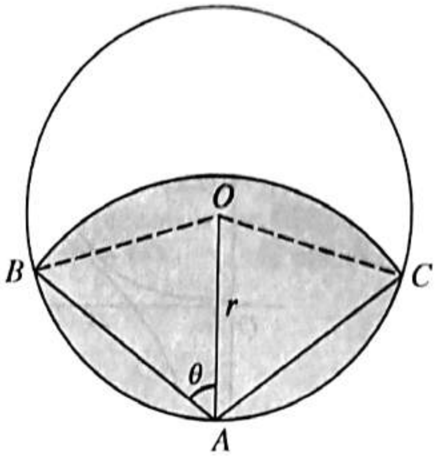
(ii) Show that if a sequence of positive values given by the iterative formula

$$x_{n+1} = \tfrac{1}{2}\ln\left(1 + \frac{10}{x_n}\right)$$

converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

20. O/N 13/P32/Q6



In the diagram, A is a point on the circumference of a circle with centre O and radius r. A circular arc with centre A meets the circumference at B and C. The angle OAB is θ radians. The shaded region is bounded by the circumference of the circle and the arc with centre A joining B and C. The area of the shaded region is equal to half the area of the circle.

(i) Show that $\cos 2\theta = \dfrac{2\sin 2\theta - \pi}{4\theta}$. [5]

(ii) Use the iterative formula

$$\theta_{n+1} = \tfrac{1}{2}\cos^{-1}\left(\frac{2\sin 2\theta_n - \pi}{4\theta_n}\right),$$

with initial value $\theta_1 = 1$, to determine θ correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]

21. M/J 13/P32/Q2

The sequence of values given by the iterative formula

$$x_{n+1} = \frac{x_n(x_n^3 + 100)}{2(x_n^3 + 25)},$$

with initial value $x_1 = 3.5$, converges to α.

(i) Use this formula to calculate α correct to 4 decimal places, showing the result of each iteration to 6 decimal places. [3]

(ii) State an equation satisfied by α and hence find the exact value of α. [2]

22. M/J 13/P31/Q10

Liquid is flowing into a small tank which has a leak. Initially the tank is empty and, t minutes later, the volume of liquid in the tank is V cm^3. The liquid is flowing into the tank at a constant rate of 80 cm^3 per minute. Because of the leak, liquid is being lost from the tank at a rate which, at any instant, is equal to kV cm^3 per minute where k is a positive constant.

(i) Write down a differential equation describing this situation and solve it to show that

$$V = \frac{1}{k}(80 - 80e^{-kt}).$$

[7]

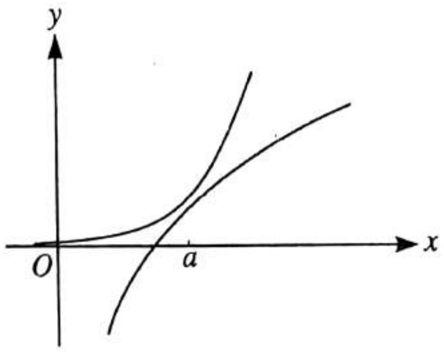
(ii) It is observed that $V = 500$ when $t = 15$, so that k satisfies the equation

$$k = \frac{4 - 4e^{-15k}}{25}.$$

Use an iterative formula, based on this equation, to find the value of k correct to 2 significant figures. Use an initial value of $k = 0.1$ and show the result of each iteration to 4 significant figures. [3]

(iii) Determine how much liquid there is in the tank 20 minutes after the liquid started flowing, and state what happens to the volume of liquid in the tank after a long time. [2]

23. M/J 13/P33/Q6



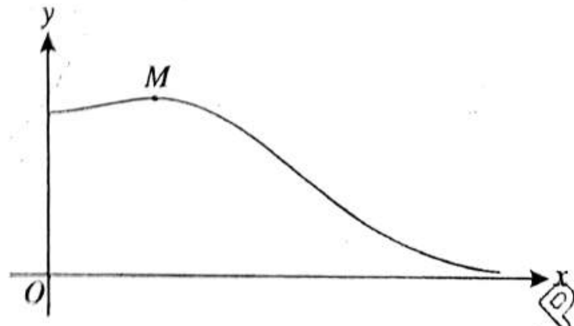
The diagram shows the curves $y = e^{2x-3}$ and $y = 2\ln x$. When $x = a$ the tangents to the curves are parallel.

(i) Show that a satisfies the equation $a = \frac{1}{2}(3 - \ln a)$. [3]

(ii) Verify by calculation that this equation has a root between 1 and 2. [2]

(iii) Use the iterative formula $a_{n+1} = \frac{1}{2}(3 - \ln a_n)$ to calculate a correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]

24. O/N 12/P32/Q8, O/N 12/P31/Q8



The diagram shows the curve $y = e^{-\frac{1}{2}x^2}\sqrt{(1 + 2x^2)}$ for $x \geqslant 0$, and its maximum point M.

(i) Find the exact value of the x-coordinate of M. [4]

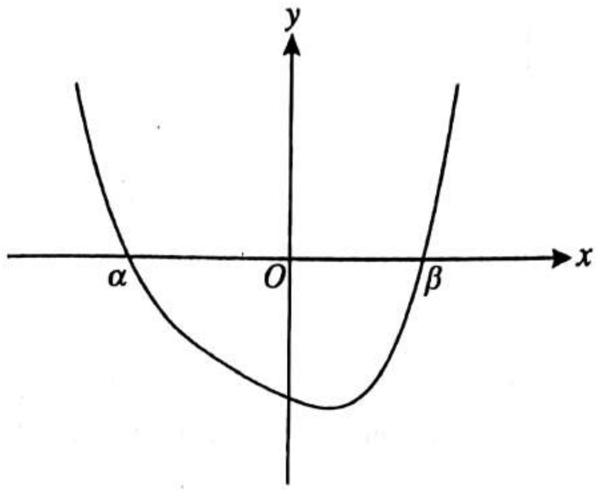
(ii) The sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{(\ln(4 + 8x_n^2))},$$

with initial value $x_1 = 2$, converges to a certain value α. State an equation satisfied by α and hence show that α is the x-coordinate of a point on the curve where $y = 0.5$. [3]

(iii) Use the iterative formula to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

25. O/N 12/P33/Q6



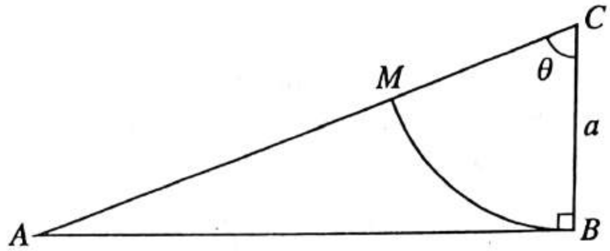
The diagram shows the curve $y = x^4 + 2x^3 + 2x^2 - 4x - 16$, which crosses the x-axis at the points $(\alpha, 0)$ and $(\beta, 0)$ where $\alpha < \beta$. It is given that α is an integer.

(i) Find the value of α. [2]

(ii) Show that β satisfies the equation $x = \sqrt[3]{(8 - 2x)}$. [3]

(iii) Use an iteration process based on the equation in part (ii) to find the value of β correct to 2 decimal places. Show the result of each iteration to 4 decimal places. [3]

26. M/J 12/P32/Q2



In the diagram, ABC is a triangle in which angle ABC is a right angle and $BC = a$. A circular arc, with centre C and radius a, joins B and the point M on AC. The angle ACB is θ radians. The area of the sector CMB is equal to one third of the area of the triangle ABC.

(i) Show that θ satisfies the equation

$$\tan\theta = 3\theta.$$ [2]

(ii) This equation has one root in the interval $0 < \theta < \frac{1}{2}\pi$. Use the iterative formula

$$\theta_{n+1} = \tan^{-1}(3\theta_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

27. M/J 12/P31/Q10

(i) It is given that $2\tan 2x + 5\tan^2 x = 0$. Denoting $\tan x$ by t, form an equation in t and hence show that either $t = 0$ or $t = \sqrt[3]{(t + 0.8)}$. [4]

(ii) It is given that there is exactly one real value of t satisfying the equation $t = \sqrt[3]{(t + 0.8)}$. Verify by calculation that this value lies between 1.2 and 1.3. [2]

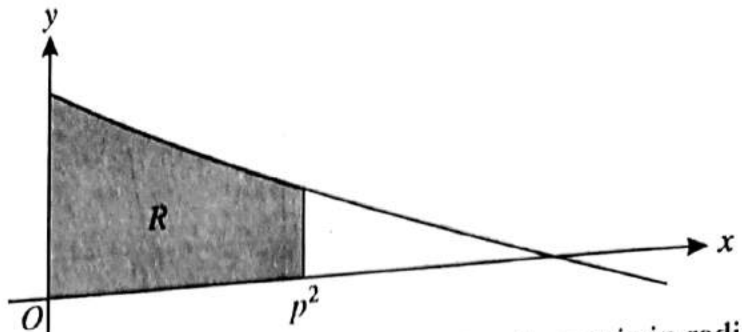
(iii) Use the iterative formula $t_{n+1} = \sqrt[3]{(t_n + 0.8)}$ to find the value of t correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iv) Using the values of t found in previous parts of the question, solve the equation

$$2\tan 2x + 5\tan^2 x = 0$$

for $-\pi \leqslant x \leqslant \pi$. [3]

28. M/J 12/P33/Q7



The diagram shows part of the curve $y = \cos(\sqrt{x})$ for $x \geqslant 0$, where x is in radians. The shaded region between the curve, the axes and the line $x = p^2$, where $p > 0$, is denoted by R. The area of R is equal to 1.

(i) Use the substitution $x = u^2$ to find $\int_0^{p^2} \cos(\sqrt{x})\,dx$. Hence show that $\sin p = \dfrac{3 - 2\cos p}{2p}$. [6]

(ii) Use the iterative formula $p_{n+1} = \sin^{-1}\left(\dfrac{3 - 2\cos p_n}{2p_n}\right)$, with initial value $p_1 = 1$, to find the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

29. O/N 11/P32/Q5

(i) By sketching a suitable pair of graphs, show that the equation

$$\sec x = 3 - \tfrac{1}{2}x^2,$$

where x is in radians, has a root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 1 and 1.4. [2]

(iii) Show that this root also satisfies the equation

$$x = \cos^{-1}\left(\frac{2}{6 - x^2}\right).$$ [1]

(iv) Use an iterative formula based on the equation in part **(iii)** to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

30. O/N 11/P31/Q5

(i) By sketching a suitable pair of graphs, show that the equation

$$\sec x = 3 - \tfrac{1}{2}x^2,$$

where x is in radians, has a root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 1 and 1.4. [2]

(iii) Show that this root also satisfies the equation

$$x = \cos^{-1}\left(\frac{2}{6 - x^2}\right).$$ [1]

(iv) Use an iterative formula based on the equation in part **(iii)** to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

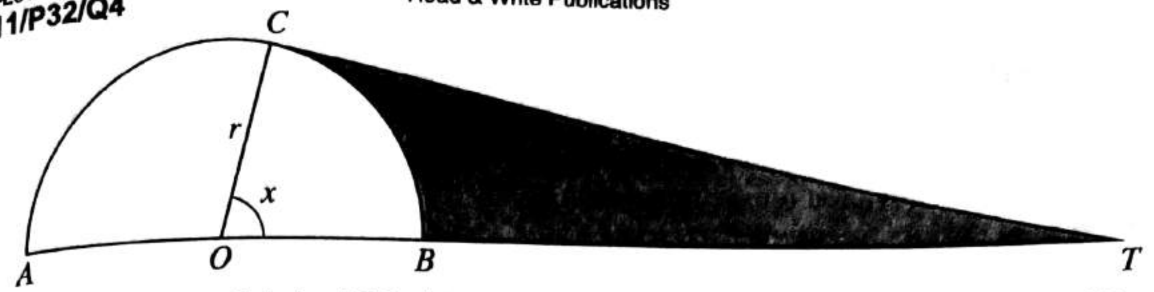
31. O/N 11/P33/Q5

It is given that $\int_1^a x \ln x\,dx = 22$, where a is a constant greater than 1.

(i) Show that $a = \sqrt{\left(\dfrac{87}{2\ln a - 1}\right)}$. [5]

(ii) Use an iterative formula based on the equation in part **(i)** to find the value of a correct to 2 decimal places. Use an initial value of 6 and give the result of each iteration to 4 decimal places. [3]

32. M/J 11/P32/Q4



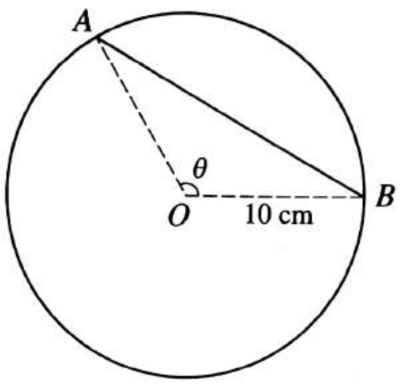
The diagram shows a semicircle ACB with centre O and radius r. The tangent at C meets AB produced at T. The angle BOC is x radians. The area of the shaded region is equal to the area of the semicircle.

(i) Show that x satisfies the equation

$$\tan x = x + \pi.$$ [3]

(ii) Use the iterative formula $x_{n+1} = \tan^{-1}(x_n + \pi)$ to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

33. M/J 11/P31/Q6



The diagram shows a circle with centre O and radius 10 cm. The chord AB divides the circle into two regions whose areas are in the ratio 1 : 4 and it is required to find the length of AB. The angle AOB is θ radians.

(i) Show that $\theta = \frac{2}{5}\pi + \sin\theta$. [3]

(ii) Showing all your working, use an iterative formula, based on the equation in part **(i)**, with an initial value of 2.1, to find θ correct to 2 decimal places. Hence find the length of AB in centimetres correct to 1 decimal place. [5]

34. M/J 11/P33/Q6

(i) By sketching a suitable pair of graphs, show that the equatio n

$$\cot x = 1 + x^2,$$

where x is in radians, has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.5 and 0.8 . [2]

(iii) Use the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{1}{1 + x_n^2}\right)$$

to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

35. O/N 10/P32/Q4, O/N/P31/Q4

(i) By sketching suitable graphs, show that the equation

$$4x^2 - 1 = \cot x$$

has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.6 and 1. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tfrac{1}{2}\sqrt{(1 + \cot x_n)}$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

36. O/N 10/P32/Q7, O/N/P31/Q7

With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point P lies on the line AB and OP is perpendicular to AB.

(i) Find a vector equation for the line AB. [1]

(ii) Find the position vector of P. [4]

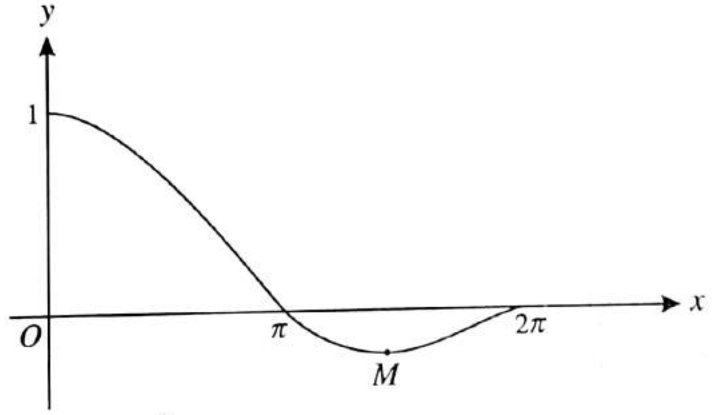
(iii) Find the equation of the plane which contains AB and which is perpendicular to the plane OAB, giving your answer in the form $ax + by + cz = d$. [4]

37. O/N 10/P33/Q7

(i) Given that $\int_1^a \frac{\ln x}{x^2}\,dx = \frac{2}{5}$, show that $a = \frac{5}{3}(1 + \ln a)$. [5]

(ii) Use an iteration formula based on the equation $a = \frac{5}{3}(1 + \ln a)$ to find the value of a correct to 2 decimal places. Use an initial value of 4 and give the result of each iteration to 4 decimal places. [3]

38. M/J 10/P32/Q4



The diagram shows the curve $y = \frac{\sin x}{x}$ for $0 < x \leqslant 2\pi$, and its minimum point M.

(i) Show that the x-coordinate of M satisfies the equation

$$x = \tan x.$$ [4]

(ii) The iterative formula

$$x_{n+1} = \tan^{-1}(x_n) + \pi$$

can be used to determine the x-coordinate of M. Use this formula to determine the x-coordinate of M correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

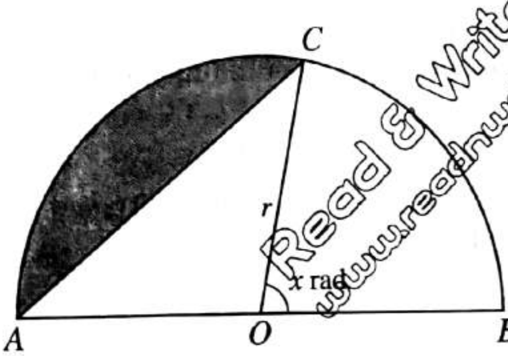
39. M/J 10/P32/Q9

The plane p has equation $3x + 2y + 4z = 13$. A second plane q is perpendicular to p and has equation $ax + y + z = 4$, where a is a constant.

(i) Find the value of a. [3]

(ii) The line with equation $\mathbf{r} = \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ meets the plane p at the point A and the plane q at the point B. Find the length of AB. [6]

40. M/J 10/P31/Q6



The diagram shows a semicircle ACB with centre O and radius r. The angle BOC is x radians. The area of the shaded segment is a quarter of the area of the semicircle.

(i) Show that x satisfies the equation

$$x = \tfrac{3}{4}\pi - \sin x.$$ [3]

(ii) This equation has one root. Verify by calculation that the root lies between 1.3 and 1.5. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tfrac{3}{4}\pi - \sin x_n$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

41. M/J 10/P31/Q10

The lines l and m have vector equations

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 4\mathbf{i} + 6\mathbf{j} + \mathbf{k} + t(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that l and m intersect. [4]

(ii) Calculate the acute angle between the lines. [3]

(iii) Find the equation of the plane containing l and m, giving your answer in the form $ax + by + cz = d$. [5]

42. M/J 10/P33/Q6

The curve $y = \dfrac{\ln x}{x+1}$ has one stationary point.

(i) Show that the x-coordinate of this point satisfies the equation

$$x = \frac{x+1}{\ln x},$$

and that this x-coordinate lies between 3 and 4. [5]

(ii) Use the iterative formula

$$x_{n+1} = \frac{x_n + 1}{\ln x_n}$$

to determine the x-coordinate correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

43. M/J 10/P33/Q10

The straight line l has equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$. The plane p has equation $3x - y + 2z = 9$. The line l intersects the plane p at the point A.

(i) Find the position vector of A. [3]

(ii) Find the acute angle between l and p. [4]

(iii) Find an equation for the plane which contains l and is perpendicular to p, giving your answer in the form $ax + by + cz = d$. [5]

44. O/N 09/P32/Q2

The equation $x^3 - 8x - 13 = 0$ has one real root.

(i) Find the two consecutive integers between which this root lies. [2]

(ii) Use the iterative formula

$$x_{n+1} = (8x_n + 13)^{\frac{1}{3}}$$

to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

45. O/N 09/P31/Q3

The sequence of values given by the iterative formula

$$x_{n+1} = \frac{3x_n}{4} + \frac{15}{x_n^3},$$

with initial value $x_1 = 3$, converges to α.

(i) Use this iterative formula to find α correct to 2 decimal places, giving the result of each iteration to 4 decimal places. [3]

(ii) State an equation satisfied by α and hence find the exact value of α. [2]

46. M/J 09/P3/Q4

The equation $x^3 - 2x - 2 = 0$ has one real root.

(i) Show by calculation that this root lies between $x = 1$ and $x = 2$. [2]

(ii) Prove that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 - 2}$$

converges, then it converges to this root. [2]

(iii) Use this iterative formula to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

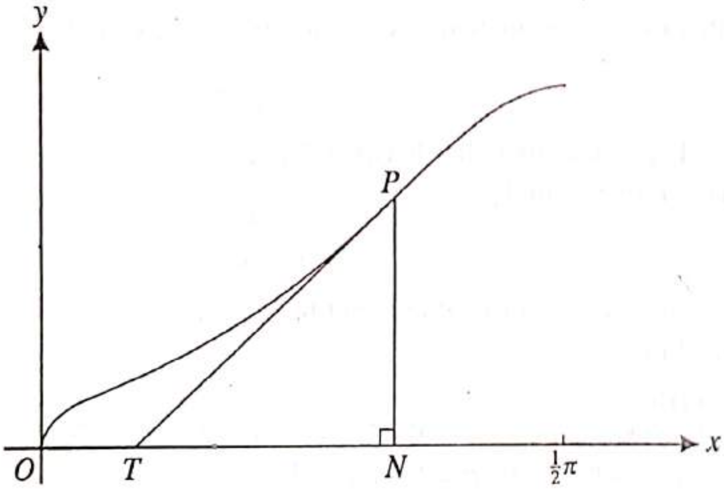
47. M/J 09/P3/Q9

The line l has equation $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$. It is given that l lies in the plane with equation $2x + by + cz = 1$, where b and c are constants.

(i) Find the values of b and c. [6]

(ii) The point P has position vector $2\mathbf{j} + 4\mathbf{k}$. Show that the perpendicular distance from P to l is $\sqrt{5}$. [5]

48. M/J 08/P3/Q3



In the diagram the tangent to a curve at a general point P with coordinates (x, y) meets the x-axis at T. The point N on the x-axis is such that PN is perpendicular to the x-axis. The curve is such that, for all values of x in the interval $0 < x < \frac{1}{2}\pi$, the area of triangle PTN is equal to $\tan x$, where x is in radians.

(i) Using the fact that the gradient of the curve at P is $\dfrac{PN}{TN}$, show that

$$\frac{dy}{dx} = \frac{1}{2}y^2 \cot x.$$ [3]

(ii) Given that $y = 2$ when $x = \frac{1}{6}\pi$, solve this differential equation to find the equation of the curve, expressing y in terms of x. [6]

49. O/N 07/P3/Q6

(i) By sketching a suitable pair of graphs, show that the equation

$$2 - x = \ln x$$

has only one root. [2]

(ii) Verify by calculation that this root lies between 1.4 and 1.7. [2]

(iii) Show that this root also satisfies the equation

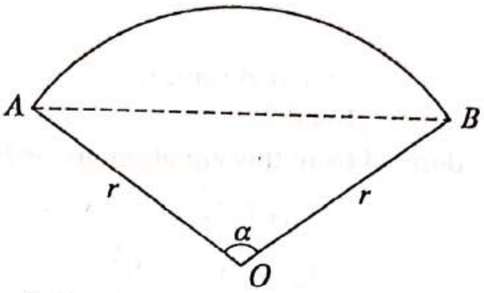
$$x = \frac{1}{3}(4 + x - 2\ln x).$$ [1]

(iv) Use the iterative formula

$$x_{n+1} = \frac{1}{3}(4 + x_n - 2\ln x_n),$$

with initial value $x_1 = 1.5$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

50. M/J 07/P3/Q6



The diagram shows a sector AOB of a circle with centre O and radius r. The angle AOB is α radians, where $0 < \alpha < \pi$. The area of triangle AOB is half the area of the sector.

(i) Show that α satisfies the equation

$$x = 2\sin x.$$

[2]

(ii) Verify by calculation that α lies between $\frac{1}{2}\pi$ and $\frac{2}{3}\pi$. [2]

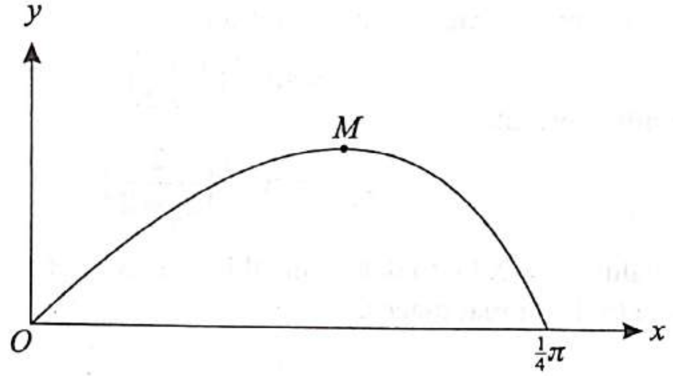
(iii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \tfrac{1}{3}(x_n + 4\sin x_n)$$

converges, then it converges to a root of the equation in part (i). [2]

(iv) Use this iterative formula, with initial value $x_1 = 1.8$, to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

51. O/N 06/P3/Q10



The diagram shows the curve $y = x\cos 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The point M is a maximum point.

(i) Show that the x-coordinate of M satisfies the equation $1 = 2x\tan 2x$. [3]

(ii) The equation in part (i) can be rearranged in the form $x = \dfrac{1}{2}\tan^{-1}\left(\dfrac{1}{2x}\right)$. Use the iterative formula

$$x_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_n}\right),$$

with initial value $x_1 = 0.4$, to calculate the x-coordinate of M correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) Use integration by parts to find the exact area of the region enclosed between the curve and the x-axis from 0 to $\frac{1}{4}\pi$. [5]

52. M/J 06/P3/Q6

(i) By sketching a suitable pair of graphs, show that the equation

$$2\cot x = 1 + e^x,$$

where x is in radians, has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.5 and 1.0. [2]

(iii) Show that this root also satisfies the equation

$$x = \tan^{-1}\left(\frac{2}{1 + e^x}\right).$$

[1]

(iv) Use the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{2}{1 + e^{x_n}}\right),$$

with initial value $x_1 = 0.7$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

53. O/N 05/P3/Q4

The equation $x^3 - x - 3 = 0$ has one real root, α.

(i) Show that α lies between 1 and 2. [2]

Two iterative formulae derived from this equation are as follows:

$$x_{n+1} = x_n^3 - 3, \qquad (A)$$

$$x_{n+1} = (x_n + 3)^{\frac{1}{3}}. \qquad (B)$$

Each formula is used with initial value $x_1 = 1.5$.

(ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [5]

54. M/J 05/P3/Q7

(i) By sketching a suitable pair of graphs, show that the equation

$$\operatorname{cosec} x = \tfrac{1}{2}x + 1,$$

where x is in radians, has a root in the interval $0 < x < \tfrac{1}{2}\pi$. [2]

(ii) Verify, by calculation, that this root lies between 0.5 and 1. [2]

(iii) Show that this root also satisfies the equation

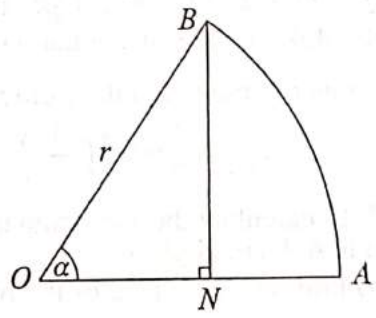
$$x = \sin^{-1}\left(\frac{2}{x+2}\right).$$ [1]

(iv) Use the iterative formula

$$x_{n+1} = \sin^{-1}\left(\frac{2}{x_n+2}\right),$$

with initial value $x_1 = 0.75$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

55. O/N 04/P3/Q7



The diagram shows a sector OAB of a circle with centre O and radius r. The angle AOB is α radians, where $0 < \alpha < \frac{1}{2}\pi$. The point N on OA is such that BN is perpendicular to OA. The area of the triangle ONB is half the area of the sector OAB.

(i) Show that α satisfies the equation $\sin 2x = x$. [3]

(ii) By sketching a suitable pair of graphs, show that this equation has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(iii) Use the iterative formula

$$x_{n+1} = \sin(2x_n),$$

with initial value $x_1 = 1$, to find α correct to 2 decimal places, showing the result of each iteration. [3]

56. M/J 04/P3/Q7

(i) The equation $x^3 + x + 1 = 0$ has one real root. Show by calculation that this root lies between -1 and 0. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 + 1}$$

converges, then it converges to the root of the equation given in part (i). [2]

(iii) Use this iterative formula, with initial value $x_1 = -0.5$, to determine the root correct to 2 decimal places, showing the result of each iteration. [3]

57. O/N 03/P3/Q5

(i) By sketching suitable graphs, show that the equation

$$\sec x = 3 - x^2$$

has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \cos^{-1}\left(\frac{1}{3 - x_n^2}\right)$$

converges, then it converges to a root of the equation given in part (i). [2]

(iii) Use this iterative formula, with initial value $x_1 = 1$, to determine the root in the interval $0 < x < \frac{1}{2}\pi$ correct to 2 decimal places, showing the result of each iteration. [3]

58. M/J 03/P3/Q8

The equation of a curve is $y = \ln x + \frac{2}{x}$, where $x > 0$.

(i) Find the coordinates of the stationary point of the curve and determine whether it is a maximum or a minimum point. [5]

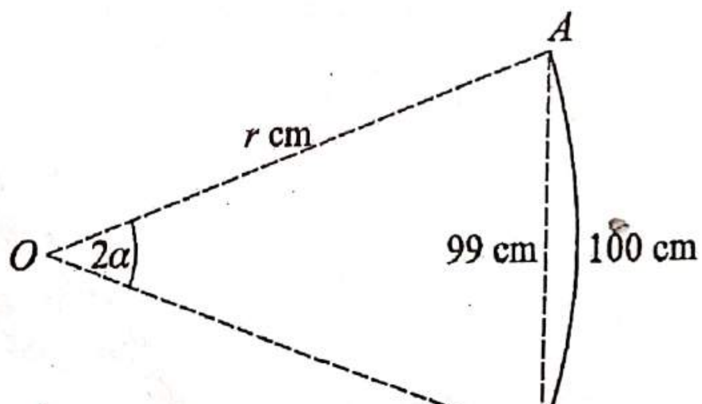
(ii) The sequence of values given by the iterative formula

$$x_{n+1} = \frac{2}{3 - \ln x_n},$$

with initial value $x_1 = 1$, converges to α. State an equation satisfied by α, and hence show that α is the x-coordinate of a point on the curve where $y = 3$. [2]

(iii) Use this iterative formula to find α correct to 2 decimal places, showing the result of each iteration. [3]

59. O/N 02/P3/Q7



The diagram shows a curved rod AB of length 100 cm which forms an arc of a circle. The end points A and B of the rod are 99 cm apart. The circle has radius r cm and the arc AB subtends an angle of 2α radians at O, the centre of the circle.

(i) Show that α satisfies the equation $\frac{99}{100}x = \sin x$ [3]

(ii) Given that this equation has exactly one root in the interval $0 < x < \frac{1}{2}\pi$, verify by calculation that this root lies between 0.1 and 0.5. [2]

(iii) Show that if the sequence of values given by the iterative formula

$$x_{n+1} = 50\sin x_n - 48.5\,x_n$$

converges, then it converges to a root of the equation in part (i). [2]

(iv) Use this iterative formula, with initial value $x_1 = 0.25$, to find α correct to 3 decimal places, showing the result of each iteration. [2]

60. M/J 02/P3/Q4

The sequence of values given by the iterative formula

$$x_{n+1} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right)$$

with initial value $x_1 = 1$, converges to α

(i) Use this formula to find α correct to 2 decimal places, showing the result of each iteration. [3]

(ii) State an equation satisfied by α. And hence find the exact value of α. [2]

Answers Section

1. M/J 18/P32/Q6

(i) Use correct method for finding the area of a segment and area of semicircle and form an equation in θ
State a correct equation in any form
Obtain the given answer correctly **3**

(ii) Calculate values of a relevant expression or pair of expressions at $\theta = 2.2$ and $\theta = 2.4$
Complete the argument correctly with correct calculated values **2**

(iii) Use $\theta_{n+1} = \frac{1}{2}\pi + \sin\theta_n$ correctly at least once
Obtain final answer 2.31
Show sufficient iterations to 4 d.p. to justify 2.31 to 2 d.p. or show there is a sign change in the interval (2.305, 2.315) **3**

2. M/J 18/P31/Q8

(i) Integrate by parts and reach $lxe^{-\frac{1}{2}x} + m\int e^{-\frac{1}{2}x}\,dx$
Obtain $-2xe^{-\frac{1}{2}x} + 2\int e^{-\frac{1}{2}x}\,dx$
Complete the integration and obtain $-2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x}$, or equivalent
Having integrated twice, use limits and equate result to 2
Obtain the given equation correctly **5**

(ii) Calculate values of a relevant expression or pair of expressions at $a = 3$ and $a = 3.5$
Complete the argument correctly with correct calculated values **2**

(iii) Use the iterative formula $a_{n+1} = 2\ln(a_n + 2)$ correctly at least once
Obtain final answer 3.36
Show sufficient iterations to 4 d.p. to justify 3.36 to 2 d.p., or show there is a sign change in the interval (3.355, 3.365) **3**

3. M/J 18/P33/Q4

(i) Use the quotient or product rule
Obtain correct derivative in any form
Equate derivative to zero and obtain the given equation **3**

(ii) Sketch a relevant graph, e.g. $y = \ln x$
Sketch a second relevant graph, e.g. $y = 1 + \frac{3}{x}$, and justify the given statement **2**

(iii) Use iterative formula $x_{n+1} = \frac{3+x}{\ln x_n}$ correctly at least once
Obtain final answer 4.97
Show sufficient iterations to 4 d.p.to justify 4.97 to 2 d.p. or show there is a sign change in the interval (4.965, 4.975) **3**

4. O/N 17/P31/Q3, O/N 17/P33/Q3

(i) Calculate value of a relevant expression or expressions at $x = 2$ and $x = 3$
Complete the argument correctly with correct calculated values **2**

(ii) Use an iterative formula correctly at least once
Show that (*B*) fails to converge
Using (*A*), obtain final answer 2.43
Show sufficient iterations to justify 2.43 to 2 d.p., or show there is a sign change in (2.425, 2.435) **4**

5. **M/J 17/P32/Q10**

(i) State or imply a correct normal vector to either plane, e.g. $\mathbf{i}+\mathbf{j}+3\mathbf{k}$ or $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$
Carry out correct process for evaluating the scalar product of two normal vectors
Using the correct process for the moduli, divide the scalar product of the two normals by the product of their moduli and evaluate the inverse cosine of the result
Obtain final answer $72.5°$ or 1.26 radians 4

(ii) *EITHER:* Substitute $y=2$ in both plane equations and solve for x or for z
Obtain $x=3$ and $z=1$

OR: Find the equation of the line of intersection of the planes
Substitute $y=2$ in line equation and solve for x or for z
Obtain $x=3$ and $z=1$

EITHER: Use scalar product to obtain an equation in a, b and c, e.g. $a+b+3c=0$
Form a second relevant equation, e.g. $2a-2b+c=0$, and solve for one ratio, e.g. $a:b$
Obtain final answer $a:b:c=7:5:-4$
Use coordinates of A and values of a, b and c in general equation and find d
Obtain answer $7x+5y-4z=27$, or equivalent

OR1: Calculate the vector product of relevant vectors, e.g. $(\mathbf{i}+\mathbf{j}+3\mathbf{k})\times(2\mathbf{i}-2\mathbf{j}+\mathbf{k})$
Obtain two correct components
Obtain correct answer, e.g. $7\mathbf{i}+5\mathbf{j}-4\mathbf{k}$
Substitute coordinates of A in plane equation with their normal and find d
Obtain answer $7x+5y-4z=27$, or equivalent

OR2: Using relevant vectors, form a two-parameter equation for the plane
State a correct equation, e.g. $\mathbf{r}=3\mathbf{i}+2\mathbf{j}+\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}+3\mathbf{k})+\mu(2\mathbf{i}-2\mathbf{j}+\mathbf{k})$
State 3 correct equations in x, y, z, λ and μ
Eliminate λ and μ
Obtain answer $7x+5y-4z=27$, or equivalent

OR3: Use the direction vector of the line of intersection of the two planes as normal vector to the plane
Two correct components
Three correct components
Substitute coordinates of A in plane equation with their normal and find d
Obtain answer $7x+5y-4z=27$, or equivalent 7

6. **M/J 17/P31/Q5**

(i) Use correct sector formula at least once and form an equation in r and x
Obtain a correct equation in any form
Rearrange in the given form 3

(ii) Calculate values of a relevant expression or expressions at $x=1$ and $x=1.5$
Complete the argument correctly with correct calculated values 2

(iii) Use the iterative formula correctly at least once
Obtain final answer 1.374
Show sufficient iterations to 5 d.p. to justify 1.374 to 3 d.p., or show there is a sign change in the interval (1.3745, 1.3755) 3

7. **M/J 17/P33/Q6 (i,ii)**

(i) State or obtain coordinates (1, 2, 1) for the mid-point of AB
Verify that the midpoint lies on m
State or imply a correct normal vector to the plane, e.g. $2\mathbf{i}+2\mathbf{j}-\mathbf{k}$
State or imply a direction vector for the segment AB, e.g. $-4\mathbf{i}-4\mathbf{j}+2\mathbf{k}$
Confirm that m is perpendicular to AB 5

(ii) State or imply that the perpendicular distance of m from the origin is $\frac{5}{3}$, or unsimplified equivalent
State or imply that n has an equation of the form $2x+2y-z=k$
Obtain answer $2x+2y-z=2$ 3

8. O/N 16/P32/Q6, O/N 16/P31/Q6

(i) Make recognizable sketch of a relevant graph
Sketch the other relevant graph and justify the given statement [2]

(ii) Use calculations to consider the value of a relevant expression at $x = 1.4$ and $x = 1.6$, or the values of relevant expressions at $x = 1.4$ and $x = 1.6$
Complete the argument correctly with correct calculated values [2]

(iii) State $x = 2\sin^{-1}\left(\dfrac{3}{x+3}\right)$
Rearrange this in the form $\operatorname{cosec}\frac{1}{2}x = \frac{1}{3}x + 1$
If working in reverse, need $\sin\dfrac{x}{2} = \left(\dfrac{3}{x+3}\right)$ for first B1 [2]

(iv) Use the iterative formula correctly at least once
Obtain final answer 1.471
Show sufficient iterations to 5 d.p. to justify 1.471 to 3 d.p., or show there is a sign change in the interval (1.4705, 1.4715) [3]

9. O/N 16/P33/Q9

(i) Differentiate both equations and equate derivatives
Obtain equation $\cos a - a\sin a = -\dfrac{k}{a^2}$
State $a\cos a = \dfrac{k}{a}$ and eliminate k
Obtain the given answer showing sufficient working [5]

(ii) Show clearly correct use of the iterative formula at least once
Obtain answer 1.077
Show sufficient iterations to 5 d.p. to justify 1.077 to 3 d.p., or show there is a sign change in the interval (1.0765, 1.0775) [3]

(iii) Use a correct method to determine k
Obtain answer $k = 0.55$ [2]

10. O/N 16/P32/Q8

(i) Use correct quotient or chain rule
Obtain correct derivative in any form
Obtain the given answer correctly [3]

(ii) State a correct equation, e.g. $-e^{-a} = -\operatorname{cosec} a \cot a$
Rearrange it correctly in the given form [2]

(iii) Calculate values of a relevant expression or pair of expressions at $x = 1$ and $x = 1.5$
Complete the argument correctly with correct calculated values [2]

(iv) Use the iterative formula correctly at least once
Obtain final answer 1.317
Show sufficient iterations to 5 d.p. to justify 1.317 to 3 d.p., or show there is a sign change in the interval (1.3165, 1,3175) [3]

11. M/J 16/P33/Q6

(i) Use the product rule
Obtain correct derivative in any form
Equate 2-term derivative to zero and obtain the given answer correctly [3]

(ii) Use calculations to consider the sign of a relevant expression at $p = 2$ and $p = 2.5$, or compare values of relevant expressions at $p = 2$ and $p = 2.5$
Complete the argument correctly with correct calculated values [2]

(iii) Use the iterative formula correctly at least once
Obtain final answer 2.15
Show sufficient iterations to 4 d.p. to justify 2.15 to 2 d.p., or show there is a sign change in the interval (2.145,2.155) [3]

12. O/N 15/P32/Q4

(i) Evaluate, or consider the sign of, $x^3 - x^2 - 6$ for two integer values of x, or equivalent
Obtain the pair $x = 2$ and $x = 3$, with no errors seen [2]

(ii) State a suitable equation, e.g. $x = \sqrt{(x + (6/x))}$
Rearrange this as $x^3 - x^2 - 6 = 0$, or work *vice versa* [2]

(iii) Use the iterative formula correctly at least once
Obtain final answer 2.219
Show sufficient iterates to 5 d.p. to justify 2.219 to 3 d.p., or show there is a sign change in the interval (2.2185, 2.2195) [3]

13. O/N 15/P33/Q4

(i) Use $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$ and equate $\frac{dy}{dx}$ to 4
Obtain $\frac{4p^3}{2p+3} = 4$ or equivalent
Confirm given result $p = \sqrt[3]{2p+3}$ correctly

(ii) Evaluate $p - \sqrt[3]{2p+3}$ or $p^3 - 2p - 3$ or equivalent at 1.8 and 2.0
Justify result with correct calculations and argument
(–0.076 and 0.087 or –0.77 and 1 respectively) [2]

(iii) Use the iterative process correctly at least once with $1.8 \leqslant p_n \leqslant 2.0$
Obtain final answer 1.89
Show sufficient iterations to at least 4 d.p. to justify 1.89 or show sign change in interval (1.885, 1.895)

14. O/N 15/P31/Q4

(i) Evaluate, or consider the sign of, $x^3 - x^2 - 6$ for two integer values of x, or equivalent
Obtain the pair $x = 2$ and $x = 3$, with no errors seen [2]

(ii) State a suitable equation, e.g. $x = \sqrt{(x + (6/x))}$
Rearrange this as $x^3 - x^2 - 6 = 0$, or work *vice versa* [2]

(iii) Use the iterative formula correctly at least once
Obtain final answer 2.219
Show sufficient iterates to 5 d.p. to justify 2.219 to 3 d.p., or show there is a sign change in the interval (2.2185, 2.2195) [3]

15. M/J 15/P32/Q5

(i) State or imply $AT = r\tan x$ or $BT = r\tan x$
Use correct arc formula and form an equation in r and x
Rearrange in the given form [3]

(ii) Calculate values of a relevant expression or expressions at $x = 1$ and $x = 1.3$
Complete the argument correctly with correct calculated values [2]

(iii) Use the iterative formula correctly at least once
Obtain final answer 1.11
Show sufficient iterations to 4 d.p. to justify 1.11 to 2 d.p., or show there is a sign change in the interval (1.105, 1.115) [3]

16. **O/N 14/P33/Q9**

(i) Sketch increasing curve with correct curvature passing through origin, for $x \ge 0$
Recognisable sketch of $y = 40 - x^3$, with equation stated, for $x > 0$
Indicate in some way the one intersection, dependent on both curves being roughly correct and both existing for some $x < 0$ [3]

(ii) Consider signs of $x^3 + \ln(x+1) - 40$ at 3 and 4 or equivalent or compare values of relevant expressions for $x = 3$ and $x = 4$
Complete argument correctly with correct calculations (–11.6 and 25.6) [2]

(iii) Use the iterative formula correctly at least once
Obtain final answer 3.377
Show sufficient iterations to justify accuracy to 3 d.p. or show sign change in interval (3.3765, 3.3775) [3]

(iv) Attempt value of $\ln(x+1)$
Obtain 1.48 [2]

17. **M/J 14/P32/Q6**

(i) Use correct arc formula and form an equation in r and x
Obtain a correct equation in any form
Rearrange in the given form 3

(ii) Consider sign of a relevant expression at $x = 1$ and $x = 1.5$, or compare values of relevant expressions at $x = 1$ and $x = 1.5$
Complete the argument correctly with correct calculated values 2

(iii) Use the iterative formula correctly at least once
Obtain final answer 1.21
Show sufficient iterations to 4 d.p. to justify 1.21 to 2 d.p., or show there is a sign change in the interval (1.205,1.215) 3

18. **M/J 14/P31/Q8**

(i) Sketch $y = \operatorname{cosec} x$ for at least $0 , x , \pi$
Sketch $y = x(\pi - x)$ for at least $0 , x , \pi$
Justify statement concerning two roots, with evidence of 1 and $\frac{1}{4}\pi^2$ for y-values on graph via scales [3]

(ii) Use $\operatorname{cosec} x = \dfrac{1}{\sin x}$ and commence rearrangement
Obtain given equation correctly, showing sufficient detail [2]

(iii) (a) Use the iterative formula correctly at least once
Obtain final answer 0.66
Show sufficient iterations to 4 decimal places to justify answer or show a sign change in the interval (0.655, 0.665) [3]

(b) Obtain 2.48 [1]

19. **M/J 14/P33/Q4**

(i) Consider sign of $x - 10/(e^{2x} - 1)$ at $x = 1$ and $x = 2$
Complete the argument correctly with correct calculated values 2

(ii) State or imply $\alpha = \frac{1}{2}\ln(1 + 10/\alpha)$
Rearrange this as $\alpha = 10/(e^{2\alpha} - 1)$ or work *vice versa* 2

(iii) Use the iterative formula correctly at least once
Obtain final answer 1.14
Show sufficient iterations to 4 d.p. to justify 1.14 to 2 d.p., or show there is a sign change in the interval (1.135, 1.145) 3

20. O/N 13/P32/Q6

(i) State or imply $AB = 2r\cos\theta$ or $AB^2 = 2r^2 - 2r^2\cos(\pi - 2\theta)$
Use correct formula to express the area of sector ABC in terms of r and θ
Use correct area formulae to express the area of a segment in terms of r and θ
State a correct equation in r and θ in any form
Obtain the given answer
[SR: If the complete equation is approached by adding two sectors to the shaded area above BO and OC give the first M1 as on the scheme, and the second M1 for using correct area formulae for a triangle AOB or AOC, and a sector AOB or AOC.] [5]

(ii) Use the iterative formula correctly at least once
Obtain final answer 0.95
Show sufficient iterations to 4 d.p. to justify 0.95 to 2 d.p., or show there is a sign change in the interval (0.945, 0.955) [3]

21. M/J 13/P32/Q2

(i) Use the iterative formula correctly at least once
Obtain final answer 3.6840
Show sufficient iterations to at least 6 d.p. to justify 3.6840, or show there is a sign change in the interval (3.68395, 3.68405) [3]

(ii) State a suitable equation, e.g. $x = \dfrac{x(x^3+100)}{2(x^3+25)}$

State that the value of α is $\sqrt[3]{50}$, or exact equivalent [2]

22. M/J 13/P31/Q10

(i) State $\frac{dV}{dt} = 80 - kV$
Correctly separate variables and attempt integration of one side
Obtain $a\ln(80 - kV) = t$ or equivalent
Obtain $-\frac{1}{k}\ln(80 - kV) = t$ or equivalent
Use $t = 0$ and $V = 0$ to find constant of integration or as limits
Obtain $-\frac{1}{k}\ln(80 - kV) = t - \frac{1}{k}\ln 80$ or equivalent
Obtain given answer $V = \frac{1}{k}(80 - 80e^{-kt})$ correctly [7]

(ii) Use iterative formula correctly at least once
Obtain final answer 0.14
Show sufficient iterations to 4 s.f. to justify answer to 2 s.f. or show a sign change in the interval (0.135, 0.145) [3]

(iii) State a value between 530 and 540 cm^3 inclusive
State or imply that volume approaches 569 cm^3 (allowing any value between 567 and 571 inclusive) [2]

23. M/J 13/P33/Q6

(i) State the correct derivatives $2e^{2x-3}$ and $2/x$
Equate derivatives and use a law of logarithms on an equation equivalent to $ke^{2x-3} = m/x$
Obtain the given result correctly (or work *vice versa*) [3]

(ii) Consider the sign of $a - \dfrac{1}{2}(3 - \ln a)$ when $a = 1$ and $a = 2$, or equivalent
Complete the argument with correct calculated values [2]

(iii) Use the iterative formula correctly at least once
Obtain final answer 1.35
Show sufficient iterations to 4 d.p. to justify 1.35 to 2 d.p., or show there is a sign change in the interval (1.345, 1.355) [3]

24. O/N 12/P32/Q8, O/N 12/P31/Q8

(i) Use correct product or quotient rule and use chain rule at least once
Obtain derivative in any correct form
Equate derivative to zero and solve an equation with at least two non-zero terms for real x
Obtain answer $x = \frac{1}{\sqrt{2}}$, or exact equivalent [4]

(ii) State a suitable equation, e.g. $\alpha = \sqrt{(\ln(4 + 8\alpha^2))}$
Rearrange to reach $e^{\alpha^2} = 4 + 8\alpha^2$
Obtain $\frac{1}{2} = e^{-\frac{1}{2}\alpha^2}\sqrt{(1 + 2\alpha^2)}$, or work *vice versa* [3]

(iii) Use the iterative formula correctly at least once
Obtain final answer 1.86
Show sufficient iterations to 4 d.p. to justify 1.86 to 2 d.p., or show there is a sign change in the interval (1.855, 1.865) [3]

25. O/N 12/P33/Q6

(i) Find y for $x = -2$
Obtain 0 and conclude that $\alpha = -2$ [2]

(ii) Either Find cubic factor by division or inspection or equivalent
Obtain $x^3 + 2x - 8$
Rearrange to confirm given equation $x = \sqrt[3]{8 - 2x}$
Or Derive cubic factor from given equation and form product with $(x - \alpha)$
$(x+2)(x^3 + 2x - 8)$
Obtain quartic $x^4 + 2x^3 + 2x^2 - 4x - 16\ (= 0)$
Or Derive cubic factor from given equation and divide the quartic by the cubic
$(x^4 + 2x^3 + 2x^2 - 4x - 16) \div (x^3 + 2x - 8)$
Obtain correct quotient and zero remainder [3]

(iii) Use the given iterative formula correctly at least once
Obtain final answer 1.67
Show sufficient iterations to at least 4 d.p. to justify answer 1.67 to 2 d.p. or show there is a change of sign in interval (1.665, 1.675) [3]

26. M/J 12/P32/Q2

(i) Using the formulae $\frac{1}{2}r^2\theta$ and $\frac{1}{2}bh$, form an equation an a and θ
Obtain given answer [2]

(ii) Use the iterative formula correctly at least once
Obtain answer $\theta = 1.32$
Show sufficient iterations to 4 d.p. to justify 1.32 to 2 d.p., or show there is a sign change in the interval (1.315, 1.325) [3]

27. M/J 12/P31/Q10

(i) Use correct identity for $\tan 2x$ and obtains $at^4 + bt^3 + ct^2 + dt = 0$, where b may be zero
Obtain correct horizontal equation, e.g. $4t + 5t^2 - 5t^4 = 0$
Obtain $kt(t^3 + et + f) = 0$ or equivalent
Confirm given results $t = 0$ and $t = \sqrt[3]{t + 0.8}$ [4]

(ii) Consider sign of $t - \sqrt[3]{t + 0.8}$ at 1.2 and 1.3 or equivalent
Justify the given statement with correct calculations (−0.06 and 0.02) [2]

(iii) Use the iterative formula correctly at least once with $1.2 < t_n < 1.3$
Obtain final answer 1.276
Show sufficient iterations to justify answer or show there is a change of sign in interval (1.2755, 1.2765) [3]

(iv) Evaluate $\tan^{-1}$ (answer from part (iii)) to obtain at least one value
Obtain -2.24 and 0.906
State $-\pi$, 0 and π
[SR If A0, B0, allow B1 for any 3 roots] [3]

28. M/J 12/P33/Q7

(i) Substitute for x and dx throughout the integral
Obtain $\int 2u \cos u \, du$
Integrate by parts and obtain answer of the form $au \sin u + b \cos u$, where $ab \neq 0$
Obtain $2u \sin u + 2 \cos u$
Use limits $u = 0$, $u = p$ correctly and equate result to 1
Obtain the given answer [6]

(ii) Use the iterative formula correctly at least once
Obtain final answer $p = 1.25$
Show sufficient iterations to 4 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in the interval (1.245, 1.255) [3]

29. O/N 11/P32/Q5

(i) Make recognisable sketch of a relevant graph over the given interval
Sketch the other relevant graph and justify the given statement [2]

(ii) Consider the sign of $\sec x - (3 - \frac{1}{2}x^2)$ at $x = 1$ and $x = 1.4$, or equivalent
Complete the argument with correct calculated values [2]

(iii) Convert the given equation to $\sec x = 3 - \frac{1}{2}x^2$ or work *vice versa* [1]

(iv) Use a correct iterative formula correctly at least once
Obtain final answer 1.13
Show sufficient iterations to 4 d.p. to justify 1.13 to 2 d.p., or show there is a sign change in the interval (1.125, 1.135)
[SR: Successive evaluation of the iterative function with $x = 1, 2, \ldots$ scores M0.] [3]

30. O/N 11/P31/Q5

(i) Make recognisable sketch of a relevant graph over the given interval
Sketch the other relevant graph and justify the given statement [2]

(ii) Consider the sign of $\sec x - (3 - \frac{1}{2}x^2)$ at $x = 1$ and $x = 1.4$, or equivalent
Complete the argument with correct calculated values [2]

(iii) Convert the given equation to $\sec x = 3 - \frac{1}{2}x^2$ or work *vice versa* [1]

(iv) Use a correct iterative formula correctly at least once
Obtain final answer 1.13
Show sufficient iterations to 4 d.p. to justify 1.13 to 2 d.p., or show there is a sign change in the interval (1.125, 1.135)
[SR: Successive evaluation of the iterative function with $x = 1, 2, \ldots$ scores M0.] [3]

31. O/N 11/P33/Q5

(i) **Either**
Use integration by parts and reach an expression $kx^2 \ln x \pm n \int x^2 . \frac{1}{x} dx$
Obtain $\frac{1}{2}x^2 \ln x - \int \frac{1}{2}x \, dx$ or equivalent
Obtain $\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$
Or
Use Integration by parts and reach an expression $kx(x\ln x - x) \pm m \int x\ln x - x dx$
Obtain $I = (x^2 \ln x - x^2) - I + \int x dx$
Obtain $\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$
Substitute limits correctly and equate to 22, having integrated twice
Rearrange and confirm given equation $a = \sqrt{\dfrac{87}{2\ln a - 1}}$ [5]

(ii) Use iterative process correctly at least once
Obtain final answer 5.86
Show sufficient iterations to 4 d.p. to justify 5.86 or show a sign change in the interval (5.855, 5.865)
($6 \to 5.8030 \to 5.8795 \to 5.8491 \to 5.8611 \to 5.8564$) **[3]**

32. M/J 11/P32/Q4
(i) State or imply $CT = r\tan x$ or $OT = r\sec x$, or equivalent **[3]**
Using correct area formulae, form an equation in r and x
Obtain the given answer correctly
(ii) Use the iterative formula correctly at least once
Obtain the final answer 1.35
Show sufficient iterations to 4 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in the interval (1.345, 1.355) **[3]**

33. M/J 11/P31/Q6
(i) State or imply area of segment is $\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$ or $50\theta - 50\sin\theta$
Attempt to form equation from area of segment = $\frac{1}{5}$ of area of circle, or equivalent
Confirm given result $\theta = \frac{2}{5}\pi + \sin\theta$ **[3]**
(ii) Use iterative formula correctly at least once
Obtain value for θ of 2.11
Show sufficient iterations to justify value of θ or show sign change in interval (2.105, 2.115)
Use correct trigonometry to find an expression for the length of AB
e.g. $20\sin 1.055$ or $\sqrt{200 - 200\cos 2.11}$
Hence 17.4 **[5]**
$[2.1 \to 2.1198 \to 2.1097 \to 2.1149 \to 2.1122]$

34. M/J 11/P33/Q6
(i) Make recognisable sketch of a relevant graph over the given range
Sketch the other relevant graph and justify the given statement **[2]**
(ii) Consider the sign of $\cot x - (1 + x^2)$ at $x = 0.5$ and $x = 0.8$, or equivalent
Complete the argument with correct calculated values **[2]**
(iii) Use the iterative formula correctly at least once with $0.5 \le x_n \le 0.8$
Obtain final answer 0.62
Show sufficient iterations to 4 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in the interval (0.615, 0.625) **[3]**

35. O/N 10/P32/Q4, O/N/P31/Q4
(i) Make recognisable sketch of a relevant graph over the given range
Sketch the other relevant graph on the same diagram and justify the given statement **[2]**
(ii) Consider sign of $4x^2 - 1 - \cot x$ at $x = 0.6$ and $x = 1$, or equivalent
Complete the argument correctly with correct calculated values **[2]**
(iii) Use the iterative formula correctly at least once
Obtain final answer 0.73
Show sufficient iterations to at least 4 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in the interval (0.725, 0.735) **[3]**

36. O/N 10/P32/Q7, O/N/P31/Q7
(i) State correct equation in any form, e.g. $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ **[1]**
(ii) *EITHER*: Equate a relevant scalar product to zero and form an equation in λ
OR 1: Equate derivative of OP^2 (or OP) to zero and form an equation in λ
OR 2: Use Pythagoras in OAP or OBP and form an equation in λ
State a correct equation in any form
Solve and obtain $\lambda = -\frac{1}{6}$ or equivalent
Obtain final answer $\overrightarrow{OP} = \frac{2}{3}\mathbf{i} + \frac{5}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}$, or equivalent **[4]**

(iii) *EITHER*: State or imply $\overrightarrow{OP}$ is a normal to the required plane
State normal vector $2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$, or equivalent
Substitute coordinates of a relevant point in $2x + 5y + 7z = d$ and evaluate d
Obtain answer $2x + 5y + 7z = 26$, or equivalent

OR 1: Find a vector normal to plane AOB and calculate its vector product with a direction vector for the line AB
Obtain answer $2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$, or equivalent
Substitute coordinates of a relevant point in $2x + 5y + 7z = d$ and evaluate d
Obtain answer $2x + 5y + 7z = 26$, or equivalent

OR 2: Set up and solve simultaneous equations in a, b, c derived from zero scalar products of $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ with (i) a direction vector for line AB, (ii) a normal to plane OAB
Obtain $a : b : c = 2 : 5 : 7$, or equivalent
Substitute coordinates of a relevant point in $2x + 5y + 7z = d$ and evaluate d
Obtain answer $2x + 5y + 7z = 26$, or equivalent

OR 3: With $Q\ (x, y, z)$ on plane, use Pythagoras in OPQ to form an equation in x, y and z
Form a correct equation
Reduce to linear form
Obtain answer $2x + 5y + 7z = 26$, or equivalent

OR 4: Find a vector normal to plane AOB and form a 2-parameter equation with relevant vectors, e.g., $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) + \mu(8\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$
State three correct equations in x, y, z, λ and μ
Eliminate λ and μ [4]
Obtain answer $2x + 5y + 7z = 26$, or equivalent

37. O/N 10/P33/Q7

(i) Attempt integration by parts
Obtain $-x^{-1}\ln x + \int \frac{1}{x^2}\,dx$, $\frac{x\ln x - x}{x^2} + 2\int \frac{\ln x}{x^2}\,dx - 2\int \frac{1}{x^2}\,dx$ or equivalent
Obtain $-x^{-1}\ln x - x^{-1}$ or equivalent
Use limits correctly, equate to $\frac{2}{5}$ and attempt rearrangement to obtain a in terms of $\ln a$ [5]
Obtain given answer $a = \frac{5}{3}(1 + \ln a)$ correctly

(ii) Use valid iterative formula correctly at least once
Obtain final answer 3.96
Show sufficient iterations to › 4 dp to justify accuracy to 2 dp or show sign change in interval (3.955, 3.965) [3]
[4 → 3.9772 → 3.9676 → 3.9636 → 3.9619]
SR: Use of $a_{n+1} = e^{\left(\frac{3}{5}a_n - 1\right)}$ to obtain 0.50 also earns 3/3.

38. M/J 10/P32/Q4

(i) Use correct quotient or product rule
Obtain correct derivative in any form
Equate derivative to zero and solve for x [4]
Obtain the given answer correctly

(ii) Use the iterative formula correctly at least once
Obtain final answer 4.49
Show sufficient iterations to at least 4 d.p. to justify its accuracy to 2 d.p., or show that there is a sign change in the interval (4.485, 4.495) [3]

39. M/J 10/P32/Q9

(i) State or imply a correct normal vector to either plane, e.g. $3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ or $a\mathbf{i} + \mathbf{j} + \mathbf{k}$
Equate scalar product of normals to zero and obtain an equation in a, e.g. $3a + 2 + 4 = 0$
Obtain $a = -2$ **[3]**

(ii) Express general point of the line in component form, e.g. $(\lambda, 1 + 2\lambda, -1 + 2\lambda)$
Either substitute components in the equation of p and solve for λ, or substitute components and the value of a in the equation of q and solve for λ
Obtain $\lambda = 1$ for point A
Obtain $\lambda = 2$ for point B
Carry out correct process for finding the length of AB
Obtain answer $AB = 3$ **[6]**
[The second M mark is dependent on both values of λ being found by correct methods.]

40. M/J 10/P31/Q6

(i) Using the formulae $\frac{1}{2}r^2\theta$ and $\frac{1}{2}r^2 \sin\theta$, or equivalent, form an equation
Obtain a correct equation in r and x and/or $x/2$ in any form
Obtain the given equation correctly **[3]**

(ii) Consider the sign of $x - (\frac{3}{4}\pi - \sin x)$ at $x = 1.3$ and $x = 1.5$, or equivalent
Complete the argument with correct calculations **[2]**

(iii) Use the iterative formula correctly at least once
Obtain final answer 1.38
Show sufficient iterations to at least 4 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in the interval (1.375, 1.385) **[3]**

41. M/J 10/P31/Q10

(i) Express general point of l or m in component form, e.g. $(1 + s, 1 - s, 1 + 2s)$ or $(4 + 2t, 6 + 2t, 1 + t)$
Equate at least two corresponding pairs of components and solve for s or t
Obtain $s = -1$ or $t = -2$
Verify that all three component equations are satisfied **[4]**

(ii) Carry out correct process for evaluating the scalar product of the direction vectors of l and m
Using the correct process for the moduli, divide the scalar product by the product of the moduli and evaluate the inverse cosine of the result
Obtain answer 74.2° (or 1.30 radians) **[3]**

(iii) *EITHER:* Use scalar product to obtain $a - b + 2c = 0$ and $2a + 2b + c = 0$
Solve and obtain one ratio, e.g. $a : b$
Obtain $a : b : c = 5 : -3 : -4$, or equivalent
Substitute coordinates of a relevant point and values for a, b and c in general equation of plane and evaluate d
Obtain answer $5x - 3y - 4z = -2$, or equivalent

OR 1: Using two points on l and one on m, or *vice versa*, state three equations in a, b, c and d
Solve and obtain one ratio, e.g. $a : b$
Obtain a ratio of three of the unknowns, e.g. $a : b : c = -5 : 3 : 4$
Use coordinates of a relevant point and found ratio to find the fourth unknown, e.g. d
Obtain answer $-5x + 3y + 4z = 2$, or equivalent

OR 2: Form a correct 2-parameter equation for the plane, e.g. $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \mu(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
State three equations in x, y, z, λ and μ
State three correct equations
Eliminate λ and μ
Obtain answer $5x - 3y - 4z = -2$, or equivalent

OR 3: Attempt to calculate vector product of direction vectors of l and m
Obtain two correct components of the product
Obtain correct product, e.g. $-5\mathbf{i}+3\mathbf{j}+4\mathbf{k}$
Form a plane equation and use coordinates of a relevant point to calculate d
Obtain answer $-5x+3y+4z=2$, or equivalent [5]

42. M/J 10/P33/Q6

(i) Use correct quotient or product rule
Obtain correct derivative in any form, e.g. $\frac{1}{x(x+1)}-\frac{\ln x}{(x+1)^2}$
Equate derivative to zero and obtain the given equation correctly
Consider the sign of $x-\frac{(x+1)}{\ln x}$ at $x=3$ and $x=4$, or equivalent
Complete the argument with correct calculated values [5]

(ii) Use the iterative formula correctly at least once, using or reaching a value in the interval (3, 4)
Obtain final answer 3.59
Show sufficient iterations to at least 4 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in the interval (3.585, 3.595) [3]

43. M/J 10/P33/Q10

(i) Express general point of the line in component form, e.g. $(2+\lambda, -1+2\lambda, -4+2\lambda)$
Substitute in plane equation and solve for λ
Obtain position vector $4\mathbf{i}+3\mathbf{j}$, or equivalent [3]

(ii) State or imply a correct vector normal to the plane, e.g. $3\mathbf{i}-\mathbf{j}+2\mathbf{k}$
Using the correct process, evaluate the scalar product of a direction vector for l and a normal for p
Using the correct process for the moduli, divide the scalar product by the product of the moduli and evaluate the inverse cosine or inverse sine of the result
Obtain answer 26.5° (or 0.462 radians) [4]

(iii) *EITHER*: State $a+2b+2c=0$ or $3a-b+2c=0$
Obtain two relevant equations and solve for one ratio, e.g. $a:b$
Obtain $a:b:c=6:4:-7$, or equivalent
Substitute coordinates of a relevant point in $6x+4y-7z=d$ and evaluate d
Obtain answer $6x+4y-7z=36$, or equivalent

*OR*1: Attempt to calculate vector product of relevant vectors, e.g. $(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\times(3\mathbf{i}-\mathbf{j}+2\mathbf{k})$
Obtain two correct components of the product
Obtain correct product, e.g. $6\mathbf{i}+4\mathbf{j}-7\mathbf{k}$
Substitute coordinates of a relevant point in $6x+4y-7z=d$ and evaluate d
Obtain answer $6x+4y-7z=36$, or equivalent

*OR*2: Attempt to form 2-parameter equation with relevant vectors
State a correct equation, e.g. $\mathbf{r}=2\mathbf{i}-\mathbf{j}-4\mathbf{k}+\lambda(\mathbf{i}+2\mathbf{j}+2\mathbf{k})+\mu(3\mathbf{i}-\mathbf{j}+2\mathbf{k})$
State three equations in x, y, z, λ, μ
Eliminate λ and μ
Obtain answer $6x+4y-7z=36$, or equivalent [5]

44. O/N 09/P32/Q2

(i) Evaluate, or consider the sign of, $x^3-8x-13$ for two integer values of x, or equivalent
Conclude $x=3$ and $x=4$ with no errors seen [2]

(ii) Use the iterative formula correctly at least once
Obtain final answer 3.43
Show sufficient iterations to at least 4 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in the interval (3.425, 3.435) [3]

45. O/N 09/P31/Q3

(i) Use the iterative formula correctly at least once
State final answer 2.78
Show sufficient iterations to at least 4 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in an appropriate function in (2.775, 2.785) [3]

(ii) State a suitable equation, e.g. $x = \frac{3}{4}x + \frac{15}{x^3}$
State that the exact value of α is $\sqrt[4]{60}$, or equivalent [2]

46. M/J 09/P3/Q4

(i) Compare signs of $x^3 - 2x - 2$ when $x = 1$ and $x = 2$, or equivalent
Complete the argument with correct calculations 2

(ii) State or imply the equation $x = (2x^3 + 2) / (3x^2 - 2)$
Rearrange this in the form $x^3 - 2x - 2 = 0$, or work *vice versa* 2

(iii) Use the iterative formula correctly at least once with $x_n > 0$
Obtain final answer 1.77
Show sufficient iterations to 4 d.p. to justify its accuracy to 2 d.p.,
or show there is a sign change
In the interval (1.765, 1.775) 3

47. M/J 09/P3/Q9

(i) **EITHER** Substitute coordinates of general point of l in equation of plane and equate constant terms, obtaining an equation in b and c
Obtain a correct equation, e.g. $8 + 2b - c = 1$
Equate the coefficient of t to zero, obtaining an equation in b and c
Obtain a correct equation, e.g. $4 - b - 2c = 0$

OR Substitute (4, 2, –1) in the plane equation
Obtain a correct equation in b and c, e.g. $2b - c = -7$
EITHER Find a second point on l and obtain an equation in b and c
Obtain a correct equation in b and c, e.g. $b + 2c = 4$
OR Calculate scalar product of a direction vector for l and a vector normal for the plane and equate to zero
Obtain a correct equation for b and c

Solve for b or for c
Obtain $b = -2$ and $c = 3$ 6

(ii) **EITHER** Find $\overrightarrow{PQ}$ for a point Q on l with parameter t, e.g. $4\mathbf{i} - 5\mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$
Calculate scalar product of $\overrightarrow{PQ}$ and a direction vector for l and equate to zero
Solve and obtain $t = -2$
Carry out a complete method for finding the length of $\overrightarrow{PQ}$
Obtain the given answer $\sqrt{5}$ correctly

OR 1 Calling (4, 2, –1) A, state $\overrightarrow{AP}$ (or $\overrightarrow{PA}$) in component form, e.g. $4\mathbf{i} - 5\mathbf{k}$
Calculate vector product of $\overrightarrow{AP}$ and a direction vector for l,
e.g. $(4\mathbf{i} - 5\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$
Obtain correct answer, e.g. $-5\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$
Divide modulus of the product by that of the direction vector
Obtain the given answer correctly

(ii) Consider sign of $x - 2\sin x$ at $x = \frac{1}{2}\pi$ and $x = \frac{2}{3}\pi$, or equivalent
Complete the argument correctly with appropriate calculations **2**

(iii) State or imply the equation $x = \frac{1}{3}(x + 4\sin x)$
Rearrange this as $x = 2\sin x$, or work *vice versa* **2**

(iv) Use the iterative formula correctly at least once
Obtain final answer 1.90
Show sufficient iterations to 4 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in the interval (1.895, 1.905)
[The final answer 1.9 scores A0]. **3**

51. O/N 06/P3/Q10

(i) Use product rule
Obtain correct derivative $\cos 2x - 2x\sin 2x$
Equate derivative to zero and obtain given answer correctly **3**

(ii) Use the iterative formula correctly at least once
Obtain final answer 0.43
Show sufficient iterations to at least 3 d.p. to justify its accuracy to 2 d.p., or show there is a sign change in the interval (0.425, 0.435) **3**

(iii) Attempt integration by parts and obtain $\pm kx\sin 2x \pm \int l\sin 2x\,dx$, where $k, l = \frac{1}{2}$, 1, or 2
Obtain $\frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx$
Obtain indefinite integral $\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x$
Use limits $x = 0$ and $x = \frac{1}{4}\pi$ having integrated twice
Obtain answer $\frac{1}{8}\pi - \frac{1}{4}$, or exact equivalent **5**

52. M/J 06/P3/Q6

(i) Make recognizable sketch of a relevant graph, e.g. $y = 2\cot x$
Sketch an appropriate second graph, e.g. $y = 1 + e^x$ correctly and justify the given statement **2**

(ii) Consider sign of $2\cot x - 1 - e^x$ at $x = 0.5$ and $x = 1$, or equivalent **2**
Complete the argument with appropriate calculations

(iii) Show that the given equation is equivalent to $x = \tan^{-1}\left(\frac{2}{1+e^x}\right)$, or *vice versa* **1**

(iv) Use the iterative formula correctly at least once
Obtain final answer 0.61
Show sufficient iterations to justify its accuracy to 2 d.p., or show there is a sign change in the interval (0.605, 0.615) **3**

53. O/N 05/P3/Q4

(i) Consider sign of $x^3 - x - 3$, or equivalent
Justify the given statement **[2]**

(ii) Apply an iterative formula correctly at least once, with initial value $x_1 = 1.5$
Show that (*A*) fails to converge
Show that (*B*) converges
Obtain final answer 1.67
Show sufficient iterations to justify its accuracy to 2 d.p., or show there is a sign change in the interval (1.665, 1.675) **[5]**

54. M/J 05/P3/Q7

(i) Make recognisable sketch of a relevant graph over the given range, e.g. $y = \text{cosec } x$
Sketch the other relevant graph, e.g. $y = \frac{1}{2}x + 1$, and justify the given statement 2

(ii) Consider sign of $\text{cosec } x - \frac{1}{2}x - 1$ at $x = 0.5$ and $x = 1$, or equivalent
Complete the argument correctly with appropriate calculations 2

(iii) Rearrange $\text{cosec } x = \frac{1}{2}x + 1$ in the given form, or *vice versa* 1

(iv) Use the iterative formula correctly at least once
Obtain final answer $x = 0.80$
Show sufficient iterations to at least 3 d.p. to justify its accuracy to 2 d.p., **or** show there is a sign change in the interval (0.795, 0.805) 3

55. O/N 04/P3/Q7

(i) Obtain area of ONB in terms of r and α e.g. $\frac{1}{2}r^2 \cos\alpha \sin\alpha$
Equate area of triangle in terms of r and α to $\frac{1}{2}\left(\frac{1}{2}r^2\alpha\right)$ or equivalent 3
Obtain given form, $\sin 2\alpha = \alpha$, correctly
[Allow use of OA and/or OB for r.]

(ii) Make recognisable sketch in one diagram over the given range of two suitable graphs, e.g. $y = \sin 2x$ and $y = x$ 2
State or imply link between intersections and roots and justify the given answer
[Allow a single graph and its intersection with $y = 0$ to earn full marks.]

(iii) Use the iterative formula correctly at least once
Obtain final answer 0.95
Show sufficient iterations to justify its accuracy to 2d.p., or show there is a sign change in (0.945, 0.955) 3
[SR: Allow the M mark if calculations are attempted in degree mode.]

56. M/J 04/P3/Q7

(i) Evaluate cubic when $x = -1$ and $x = 0$ 2
Justify given statement correctly
[If calculations are not given but justification uses correct statements about signs, award B1.]

(ii) State $x = \dfrac{2x^3 - 1}{3x^2 + 1}$, or equivalent 2
Rearrange this in the form $x^3 + x + 1 = 0$ (or *vice versa*)

(iii) Use the iterative formula correctly at least once
Obtain final answer -0.68
Show sufficient iterations to justify its accuracy to 2d.p., or show there is a sign change in the interval (-0.685, -0.675) 3

57. O/N 03/P3/Q5

(i) Make recognizable sketch of $y = \sec x$ or $y = 3 - x^2$, for $0 < x < \frac{1}{2}\pi$ [2]
Sketch the other graph correctly and justify the given statement
[Award B1 for a sketch with positive y-intercept and correct concavity. A correct sketch of $y = \cos x$ can only earn B1 in the presence of $1/(3 - x^2)$. Allow a correct single graph and its intersection with $y = 0$ to earn full marks.]

(ii) State or imply equation $\alpha = \cos^{-1}(1/(3 - \alpha^2))$ or $\cos\alpha = 1/(3 - \alpha^2)$ [2]
Rearrange this in the form given in part (i) i.e. $\sec\alpha = 3 - \alpha^2$
[Or work *vice versa*.]

(iii) Use the iterative formula with $0 \le x_1 \le \sqrt{2}$
Obtain final answer 1.03
Show sufficient iterations to justify its accuracy to 2d.p. or show there is a sign change in the interval (1.025, 1.035) [3]

58. M/J 03/P3/Q8

(i) State or imply $w = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi$ (allow decimals)

Obtain answer $uw = -\sqrt{3} - i$ (allow decimals)

Multiply numerator and denominator of $\frac{u}{w}$ by $-1 - i\sqrt{3}$, or equivalent

Obtain answer $\frac{u}{w} = \sqrt{3} - i$ (allow decimals) **[4]**

(ii) Show U on an Argand diagram correctly

Show *A* and *B* in relatively correct positions **[2]**

(iii) Prove that *AB* = *UA* (or *UB*), or prove that angle *AUB* = angle *ABU* (or angle *BAU*) or prove, for example, that *AO* = *OB* and angle *AOB* = 120°, or prove that one angle of triangle *UAB* equals 60°

Complete a proof that triangle *UAB* is equilateral **[2]**

59. O/N 02/P3/Q7

(i) State or obtain a relevant equation e.g. $2r\alpha = 100$

State or obtain a second independent relevant equation e.g. $2r\sin\alpha = 99$

Derive the given equation in x (or α) correctly 3

(ii) Calculate ordinates at $x = 0.1$ and $x = 0.5$ of a suitable function or pair of functions

Justify the given statement correctly 2

(If calculations are not given but the given statement is justified using correct statements about the signs of a suitable function or the difference between a pair of suitable functions, award B1.]

(iii) State $x = 50\sin x - 48.5x$, or equivalent

Rearrange this in the form given in part **(i)** (or *vice versa*) 2

(iv) Use the method of iteration at least once with $0.1 \le x_n \le 0.5$

Obtain final answer 0.245, showing sufficient iterations to justify its accuracy to 3 d.p., or showing a sign change in the interval (0.2445, 0.2455) 2

[SR: both the M marks are available if calculations are attempted in degree mode.]

60. M/J 02/P3/Q4

(i) Use the formula correctly at least once

State $\alpha = 1.26$ as final answer

Show sufficient iterations to justify $a = 1.26$ to 2 d.p. or show there is a sign change in the interval (1.255, 1.265) 3

(ii) State any suitable equation in one unknown e.g. $x = \frac{2}{3}\left(x + \frac{1}{x^2}\right)$

State exact value of α (or x) is $\sqrt[3]{2}$ or $2^{\frac{2}{3}}$ 2

UNIT 7

Vectors

A-Level
Mathematics Paper 3
Topical Workbook

READ & WRITE PUBLICATIONS
Sale Point: Shop # 25-28 Lower Ground Floor,
Haadia Haleema Centre, Ghazni Street,
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Tel: +92 42-35714038
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E-mail: readandwriteoffice@gmail.com

Unit-7: Vectors

1. M/J 18/P32/Q10

Two lines l and m have equations $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + s(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ respectively.

(i) Show that the lines are skew. [4]

A plane p is parallel to the lines l and m.

(ii) Find a vector that is normal to p. [3]

(iii) Given that p is equidistant from the lines l and m, find the equation of p. Give your answer in the form $ax + by + cz = d$. [3]

2. M/J 18/P31/Q10

The point P has position vector $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. The line l has equation $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.

(i) Find the length of the perpendicular from P to l, giving your answer correct to 3 significant figures. [5]

(ii) Find the equation of the plane containing l and P, giving your answer in the form $ax + by + cz = d$. [5]

3. M/J 18/P33/Q10

The points A and B have position vectors $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} + \mathbf{j} + \mathbf{k}$ respectively. The line l has equation $\mathbf{r} = 4\mathbf{i} + 6\mathbf{j} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$.

(i) Show that l does not intersect the line passing through A and B. [5]

The point P, with parameter t, lies on l and is such that angle PAB is equal to $120°$.

(ii) Show that $3t^2 + 8t + 4 = 0$. Hence find the position vector of P. [6]

4. O/N 17/P32/Q10

Two planes p and q have equations $x + y + 3z = 8$ and $2x - 2y + z = 3$ respectively.

(i) Calculate the acute angle between the planes p and q. [4]

(ii) The point A on the line of intersection of p and q has y-coordinate equal to 2. Find the equation of the plane which contains the point A and is perpendicular to both the planes p and q. Give your answer in the form $ax + by + cz = d$. [7]

5. O/N 17/P31/Q10, O/N 17/P33/Q10

The equations of two lines l and m are $\mathbf{r} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ respectively.

(i) Show that the lines do not intersect. [3]

(ii) Calculate the acute angle between the directions of the lines. [3]

(iii) Find the equation of the plane which passes through the point $(3, -2, -1)$ and which is parallel to both l and m. Give your answer in the form $ax + by + cz = d$. [5]

6. M/J 17/P32/Q9

Relative to the origin O, the point A has position vector given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. The line l has equation $\mathbf{r} = 9\mathbf{i} - \mathbf{j} + 8\mathbf{k} + \mu(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.

(i) Find the position vector of the foot of the perpendicular from A to l. Hence find the position vector of the reflection of A in l. [5]

(ii) Find the equation of the plane through the origin which contains l. Give your answer in the form $ax + by + cz = d$. [3]

(iii) Find the exact value of the perpendicular distance of A from this plane. [3]

7. M/J 17/P31/Q6

The plane with equation $2x + 2y - z = 5$ is denoted by m. Relative to the origin O, the points A and B have coordinates $(3, 4, 0)$ and $(-1, 0, 2)$ respectively.

(i) Show that the plane m bisects AB at right angles. [5]

A second plane p is parallel to m and nearer to O. The perpendicular distance between the planes is 1.

(ii) Find the equation of p, giving your answer in the form $ax + by + cz = d$. [3]

8. M/J 17/P33/Q10

The points A and B have position vectors given by $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$. The line l has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + m\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} - 4\mathbf{k})$, where m is a constant.

(i) Given that the line l intersects the line passing through A and B, find the value of m. [5]

(ii) Find the equation of the plane which is parallel to $\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ and contains the points A and B. Give your answer in the form $ax + by + cz = d$. [5]

9. O/N 16/P32/Q8, O/N 16/P31/Q8

Two planes have equations $3x + y - z = 2$ and $x - y + 2z = 3$.

(i) Show that the planes are perpendicular. [3]

(ii) Find a vector equation for the line of intersection of the two planes. [6]

10. O/N 16/P33/Q10

The line l has vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$.

(i) Find the position vectors of the two points on the line whose distance from the origin is $\sqrt{(10)}$. [5]

(ii) The plane p has equation $ax + y + z = 5$, where a is a constant. The acute angle between the line l and the plane p is equal to $\sin^{-1}\left(\frac{2}{3}\right)$. Find the possible values of a. [5]

11. M/J 16/P32/Q9

The points A, B and C have position vectors, relative to the origin O, given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$. A fourth point D is such that the quadrilateral $ABCD$ is a parallelogram.

(i) Find the position vector of D and verify that the parallelogram is a rhombus. [5]

(ii) The plane p is parallel to OA and the line BC lies in p. Find the equation of p, giving your answer in the form $ax + by + cz = d$. [5]

12. M/J 16/P31/Q9

With respect to the origin O, the points A, B, C, D have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \quad \overrightarrow{OC} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OD} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

(i) Find the equation of the plane containing A, B and C, giving your answer in the form $ax + by + cz = d$. [6]

(ii) The line through D parallel to OA meets the plane with equation $x + 2y - z = 7$ at the point P. Find the position vector of P and show that the length of DP is $2\sqrt{(14)}$. [5]

13. M/J 16/P33/Q8

The points A and B have position vectors, relative to the origin O, given by $\overrightarrow{OA} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{i} + 3\mathbf{k}$. The line l has vector equation $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

(i) Show that the line passing through A and B does not intersect l. [4]

(ii) Show that the length of the perpendicular from A to l is $\frac{1}{\sqrt{2}}$. [5]

14. O/N 15/P32/Q7, O/N 15/P31/Q7

The points A, B and C have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.$$

The plane m is perpendicular to AB and contains the point C.

(i) Find a vector equation for the line passing through A and B. [2]

(ii) Obtain the equation of the plane m, giving your answer in the form $ax + by + cz = d$. [2]

(iii) The line through A and B intersects the plane m at the point N. Find the position vector of N and show that $CN = \sqrt{(13)}$. [5]

15. O/N 15/P33/Q8

A plane has equation $4x - y + 5z = 39$. A straight line is parallel to the vector $\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and passes through the point A (0, 2, −8). The line meets the plane at the point B.

(i) Find the coordinates of B. [3]

(ii) Find the acute angle between the line and the plane. [4]

(iii) The point C lies on the line and is such that the distance between C and B is twice the distance between A and B. Find the coordinates of each of the possible positions of the point C. [3]

16. M/J 15/P32/Q10

The points A and B have position vectors given by $\overrightarrow{OA} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OB} = \mathbf{i} + \mathbf{j} + 5\mathbf{k}$. The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(i) Show that l does not intersect the line passing through A and B. [5]

(ii) Find the equation of the plane containing the line l and the point A. Give your answer in the form $ax + by + cz = d$. [6]

17. M/J 15/P31/Q6

The straight line l_1 passes through the points (0, 1, 5) and (2, −2, 1). The straight line l_2 has equation $\mathbf{r} = 7\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$.

(i) Show that the lines l_1 and l_2 are skew. [6]

(ii) Find the acute angle between the direction of the line l_2 and the direction of the x-axis. [3]

18. M/J 15/P33/Q9

Two planes have equations $x + 3y - 2z = 4$ and $2x + y + 3z = 5$. The planes intersect in the straight line l.

(i) Calculate the acute angle between the two planes. [4]

(ii) Find a vector equation for the line l. [6]

19. O/N 14/P32/Q10, O/N 14/P31/Q10

The line l has equation $\mathbf{r} = 4\mathbf{i} - 9\mathbf{j} + 9\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. The point A has position vector $3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$.

(i) Show that the length of the perpendicular from A to l is 15. [5]

(ii) The line l lies in the plane with equation $ax + by - 3z + 1 = 0$, where a and b are constants. Find the values of a and b. [5]

20. O/N 14/P33/Q7

The equations of two straight lines are

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = a\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 3a\mathbf{k}),$$

where a is a constant.

(i) Show that the lines intersect for all values of a. [4]

(ii) Given that the point of intersection is at a distance of 9 units from the origin, find the possible values of a. [4]

21. M/J 14/P32/Q10

Referred to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = 3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}.$$

(i) Find the exact value of the cosine of angle BAC. [4]

(ii) Hence find the exact value of the area of triangle ABC. [3]

(iii) Find the equation of the plane which is parallel to the y-axis and contains the line through B and C. Give your answer in the form $ax + by + cz = d$. [5]

22. M/J 14/P31/Q7

The straight line l has equation $\mathbf{r} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$. The plane p passes through the point $(4, -1, 2)$ and is perpendicular to l.

(i) Find the equation of p, giving your answer in the form $ax + by + cz = d$. [2]

(ii) Find the perpendicular distance from the origin to p. [3]

(iii) A second plane q is parallel to p and the perpendicular distance between p and q is 14 units. Find the possible equations of q. [3]

23. M/J 14/P33/Q10

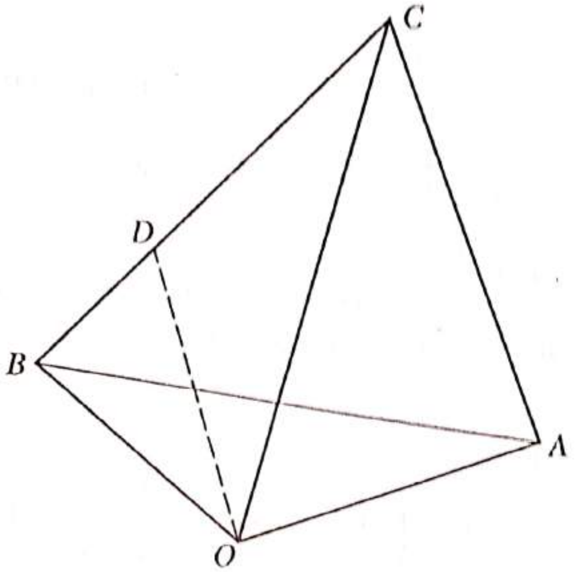
The line l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ and the plane p has equation $2x + 3y - 5z = 18$.

(i) Find the position vector of the point of intersection of l and p. [3]

(ii) Find the acute angle between l and p. [4]

(iii) A second plane q is perpendicular to the plane p and contains the line l. Find the equation of q, giving your answer in the form $ax + by + cz = d$. [5]

24. O/N 13/P32/Q9



The diagram shows three points A, B and C whose position vectors with respect to the origin O are given by $\overrightarrow{OA} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$. The point D lies on BC, between B and C, and is such that $CD = 2DB$.

(i) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$. [6]

(ii) Find the position vector of D. [1]

(iii) Show that the length of the perpendicular from A to OD is $\frac{1}{3}\sqrt{(65)}$. [4]

25. O/N 13/P33/Q6

Two planes have equations $3x - y + 2z = 9$ and $x + y - 4z = -1$.

(i) Find the acute angle between the planes. [3]

(ii) Find a vector equation of the line of intersection of the planes. [6]

26. M/J 13/P32/Q10

The points A and B have position vectors $2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ respectively. The plane p has equation $x + y = 5$.

(i) Find the position vector of the point of intersection of the line through A and B and the plane p. [4]

(ii) A second plane q has an equation of the form $x + by + cz = d$, where b, c and d are constants. The plane q contains the line AB, and the acute angle between the planes p and q is 60°. Find the equation of q. [7]

27. M/J 13/P31/Q6

The points P and Q have position vectors, relative to the origin O, given by

$$\overrightarrow{OP} = 7\mathbf{i} + 7\mathbf{j} - 5\mathbf{k} \quad \text{and} \quad \overrightarrow{OQ} = -5\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

The mid-point of PQ is the point A. The plane Π is perpendicular to the line PQ and passes through A.

(i) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

(ii) The straight line through P parallel to the x-axis meets Π at the point B. Find the distance AB, correct to 3 significant figures. [5]

28. M/J 13/P33/Q10

The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(a\mathbf{i} + 2\mathbf{j} + \mathbf{k})$, where a is a constant. The plane p has equation $x + 2y + 2z = 6$. Find the value or values of a in each of the following cases.

(i) The line l is parallel to the plane p. [2]

(ii) The line l intersects the line passing through the points with position vectors $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} - \mathbf{k}$. [4]

(iii) The acute angle between the line l and the plane p is $\tan^{-1} 2$. [5]

29. O/N 12/P32/Q10,O/N 12/P31/Q10

With respect to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}.$$

The plane m is parallel to $\overrightarrow{OC}$ and contains A and B.

(i) Find the equation of m, giving your answer in the form $ax + by + cz = d$. [6]

(ii) Find the length of the perpendicular from C to the line through A and B. [5]

30. O/N 12/P33/Q8

Two lines have equations

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} p \\ 4 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix},$$

where p is a constant. It is given that the lines intersect.

(i) Find the value of p and determine the coordinates of the point of intersection. [5]

(ii) Find the equation of the plane containing the two lines, giving your answer in the form $ax + by + cz = d$, where a, b, c and d are integers. [5]

31. M/J 12/P32/Q10

Two planes, m and n, have equations $x + 2y - 2z = 1$ and $2x - 2y + z = 7$ respectively. The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

(i) Show that l is parallel to m. [3]

(ii) Find the position vector of the point of intersection of l and n. [3]

(iii) A point P lying on l is such that its perpendicular distances from m and n are equal. Find the position vectors of the two possible positions for P and calculate the distance between them. [6]

[The perpendicular distance of a point with position vector $x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ from the plane $ax + by + cz = d$ is $\dfrac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{(a^2 + b^2 + c^2)}}$.]

32. M/J 12/P31/Q8

The point P has coordinates $(-1, 4, 11)$ and the line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

(i) Find the perpendicular distance from P to l. [4]

(ii) Find the equation of the plane which contains P and l, giving your answer in the form $ax + by + cz = d$, where a, b, c and d are integers. [5]

33. M/J 12/P33/Q9

The lines l and m have equations $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(a\mathbf{i} + b\mathbf{j} - \mathbf{k})$ respectively, where a and b are constants.

(i) Given that l and m intersect, show that

$$2a - b = 4.$$ [4]

(ii) Given also that l and m are perpendicular, find the values of a and b. [4]

(iii) When a and b have these values, find the position vector of the point of intersection of l and m. [2]

34. O/N 11/P32/Q7, O/N 11/P31/Q7

With respect to the origin O, the position vectors of two points A and B are given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point P lies on the line through A and B, and $\overrightarrow{AP} = \lambda\overrightarrow{AB}$.

(i) Show that $\overrightarrow{OP} = (1 + 2\lambda)\mathbf{i} + (2 + 2\lambda)\mathbf{j} + (2 - 2\lambda)\mathbf{k}$. [2]

(ii) By equating expressions for $\cos AOP$ and $\cos BOP$ in terms of λ, find the value of λ for which OP bisects the angle AOB. [5]

(iii) When λ has this value, verify that $AP : PB = OA : OB$. [1]

35. O/N 11/P33/Q9

The line l has equation $\mathbf{r} = \begin{pmatrix} a \\ 1 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$, where a is a constant. The plane p has equation $2x - 2y + z = 10$.

(i) Given that l does not lie in p, show that l is parallel to p. [2]

(ii) Find the value of a for which l lies in p. [2]

(iii) It is now given that the distance between l and p is 6. Find the possible values of a. [5]

36. M/J 11/P32/Q9

Two planes have equations $x + 2y - 2z = 7$ and $2x + y + 3z = 5$.

(i) Calculate the acute angle between the planes. [4]

(ii) Find a vector equation for the line of intersection of the planes. [6]

37. M/J 11/P31/Q3

Points A and B have coordinates $(-1, 2, 5)$ and $(2, -2, 11)$ respectively. The plane p passes through B and is perpendicular to AB.

(i) Find an equation of p, giving your answer in the form $ax + by + cz = d$. [3]

(ii) Find the acute angle between p and the y-axis. [4]

38. M/J 11/P33/Q10

With respect to the origin O, the lines l and m have vector equations $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ respectively.

(i) Prove that l and m do not intersect. [4]

(ii) Calculate the acute angle between the directions of l and m. [3]

(iii) Find the equation of the plane which is parallel to l and contains m, giving your answer in the form $ax + by + cz = d$. [5]

39. O/N 10/P33/Q6

The straight line l passes through the points with coordinates $(-5, 3, 6)$ and $(5, 8, 1)$. The plane p has equation $2x - y + 4z = 9$.

(i) Find the coordinates of the point of intersection of l and p. [4]

(ii) Find the acute angle between l and p. [4]

40. O/N 09/P32/Q10

The plane p has equation $2x - 3y + 6z = 16$. The plane q is parallel to p and contains the point with position vector $\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

(i) Find the equation of q, giving your answer in the form $ax + by + cz = d$. [2]

(ii) Calculate the perpendicular distance between p and q. [3]

(iii) The line l is parallel to the plane p and also parallel to the plane with equation $x - 2y + 2z = 5$. Given that l passes through the origin, find a vector equation for l. [5]

41. O/N 09/P31/Q6

With respect to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} - \mathbf{k}, \quad \overrightarrow{OB} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}.$$

The mid-point of AB is M. The point N lies on AC between A and C and is such that $AN = 2NC$.

(i) Find a vector equation of the line MN. [4]

(ii) It is given that MN intersects BC at the point P. Find the position vector of P. [4]

42. O/N 08/P03/Q7

Two planes have equations $2x - y - 3z = 7$ and $x + 2y + 2z = 0$.

(i) Find the acute angle between the planes. [4]

(ii) Find a vector equation for their line of intersection. [6]

43. M/J 08/P03/Q10

The points A and B have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

The line l has vector equation

$$\mathbf{r} = (1 - 2t)\mathbf{i} + (5 + t)\mathbf{j} + (2 - t)\mathbf{k}.$$

(i) Show that l does not intersect the line passing through A and B. [4]

(ii) The point P lies on l and is such that angle PAB is equal to 60°. Given that the position vector of P is $(1 - 2t)\mathbf{i} + (5 + t)\mathbf{j} + (2 - t)\mathbf{k}$, show that $3t^2 + 7t + 2 = 0$. Hence find the only possible position vector of P. [6]

44. O/N 07/P03/Q10

The straight line l has equation $\mathbf{r} = \mathbf{i} + 6\mathbf{j} - 3\mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$. The plane p has equation $(\mathbf{r} - 3\mathbf{i}).(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = 0$. The line l intersects the plane p at the point A.

(i) Find the position vector of A. [3]

(ii) Find the acute angle between l and p. [4]

(iii) Find a vector equation for the line which lies in p, passes through A and is perpendicular to l. [5]

45. M/J 07/P03/Q7

Let $I = \int_1^4 \frac{1}{x(4 - \sqrt{x})}\, dx$.

(i) Use the substitution $u = \sqrt{x}$ to show that $I = \int_1^2 \frac{2}{u(4 - u)}\, du$. [3]

(ii) Hence show that $I = \frac{1}{2}\ln 3$. [6]

46. O/N 06/P03/Q7

The line l has equation $\mathbf{r} = \mathbf{j} + \mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$. The plane p has equation $x + 2y + 3z = 5$.

(i) Show that the line l lies in the plane p. [3]

(ii) A second plane is perpendicular to the plane p, parallel to the line l and contains the point with position vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. Find the equation of this plane, giving your answer in the form $ax + by + cz = d$. [6]

47. M/J 06/P03/Q10

The points A and B have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}.$$

The line l passes through A and is parallel to OB. The point N is the foot of the perpendicular from B to l.

(i) State a vector equation for the line l. [1]

(ii) Find the position vector of N and show that $BN = 3$. [6]

(iii) Find the equation of the plane containing A, B and N, giving your answer in the form $ax + by + cz = d$. [5]

48. O/N 05/P03/Q10

The straight line l passes through the points A and B with position vectors

$$2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$

respectively. This line intersects the plane p with equation $x - 2y + 2z = 6$ at the point C.

(i) Find the position vector of C. [4]

(ii) Find the acute angle between l and p. [4]

(iii) Show that the perpendicular distance from A to p is equal to 2. [3]

49. M/J 05/P03/Q10

With respect to the origin O, the points A and B have position vectors given by

$$\overrightarrow{OA} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}.$$

The line l has vector equation $\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

(i) Prove that the line l does not intersect the line through A and B. [5]

(ii) Find the equation of the plane containing l and the point A, giving your answer in the form $ax + by + cz = d$. [6]

50. O/N 04/P03/Q9

The lines l and m have vector equations

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + s(\mathbf{i} + \mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that l and m do not intersect. [4]

The point P lies on l and the point Q has position vector $2\mathbf{i} - \mathbf{k}$.

(ii) Given that the line PQ is perpendicular to l, find the position vector of P. [4]

(iii) Verify that Q lies on m and that PQ is perpendicular to m. [2]

51. M/J 04/P03/Q11

With respect to the origin O, the points P, Q, R, S have position vectors given by

$$\overrightarrow{OP} = \mathbf{i} - \mathbf{k}, \quad \overrightarrow{OQ} = -2\mathbf{i} + 4\mathbf{j}, \quad \overrightarrow{OR} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OS} = 3\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}.$$

(i) Find the equation of the plane containing P, Q and R, giving your answer in the form $ax + by + cz = d$. [6]

(ii) The point N is the foot of the perpendicular from S to this plane. Find the position vector of N and show that the length of SN is 7. [6]

52. O/N 03/P03/Q10

The lines l and m have vector equations

$$\mathbf{r} = \mathbf{i} - 2\mathbf{k} + s(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 6\mathbf{i} - 5\mathbf{j} + 4\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that l and m intersect, and find the position vector of their point of intersection. [5]

(ii) Find the equation of the plane containing l and m, giving your answer in the form $ax + by + cz = d$. [6]

53. M/J 03/P03/Q9

Two planes have equations $x + 2y - 2z = 2$ and $2x - 3y + 6z = 3$. The planes intersect in the straight line l.

(i) Calculate the acute angle between the two planes. [4]

(ii) Find a vector equation for the line l. [6]

54. O/N 02/P03/Q10

With respect to the origin O, the points A, B, C, D, have position given by

$\overrightarrow{OA} = 4i + k, \quad \overrightarrow{OB} = 5i - 2j - 2k, \quad \overrightarrow{OC} = i + j, \quad \overrightarrow{OD} = -i - 4k.$

(i) Calculate the acute angle between the lines AB and CD. [4]

(ii) Prove that the lines AB and CD intersect. [4]

(iii) The point P has position vector i + 5j + 6k. Show that the perpendicular distance from P to the line AB is equal to $\sqrt{3}$. [4]

55. M/J 02/P03/Q8

The straight line l passes through the points A and B whose position vectors are i + k and 4i − j + 3k respectively. The plane p has equation x + 3y − 2z = 3.

(i) Given that l intersects p, find the position vector of the point of intersection. [4]

(ii) Find the equation of the plane which contains l and is perpendicular to p, giving your answer in the form ax + by + cz = 1. [6]

Answers Section

1. **M/J 18/P32/Q10**

(i) Equate at least two pairs of components and solve for s or for t
Obtain correct answer for s or t, e.g. $s=-6,\ t=-11$
Verify that all three equations are not satisfied and the lines fail to intersect
State that the lines are not parallel **[4]**

(ii) *EITHER* :
Use scalar product to obtain a relevant equation in a, b and c, e.g. $2a+3b-c=0$
Obtain a second equation, e.g. $a+2b+c=0$, and solve for one ratio, e.g. $a:b$
Obtain $a:b:c$ and state correct answer, e.g. $5\mathbf{i}-3\mathbf{j}+\mathbf{k}$, or equivalent

OR :
Attempt to calculate vector product of relevant vectors, e.g. $(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\times(\mathbf{i}+2\mathbf{j}+\mathbf{k})$
Obtain two correct components
Obtain correct answer, e.g. $5\mathbf{i}-3\mathbf{j}+\mathbf{k}$ **[3]**

(iii) *EITHER*:
State position vector or coordinates of the mid-point of a line segment joining points on l and m, e.g. $\frac{3}{2}\mathbf{i}+\mathbf{j}+\frac{5}{2}\mathbf{k}$
Use the result of **(ii)** and the mid-point to find d
Obtain answer $5x-3y+z=7$, or equivalent

OR :
Using the result of part **(ii)**, form an equation in d by equating perpendicular distances to the plane of a point on l and a point on m
State a correct equation, e.g. $\left|\frac{14-d}{\sqrt{35}}\right|=\left|\frac{-d}{\sqrt{35}}\right|$
Solve for d and obtain answer $5x-3y+z=7$, or equivalent **[3]**

2. **M/J 18/P31/Q10**

(i) *EITHER*
Find $\overrightarrow{PQ}$ (or $\overrightarrow{QP}$) for a general point Q on l, e.g. $(1+\mu)\mathbf{i}+(4+2\mu)\mathbf{j}+(4+3\mu)\mathbf{k}$
Calculate the scalar product of $\overrightarrow{PQ}$ and a direction vector for l and equate to zero
Solve and obtain correct solution e.g. $\mu=-\frac{3}{2}$
Carry out method to calculate PQ
Obtain answer 1.22

*OR*1:

Find $\overrightarrow{PQ}$ (or $\overrightarrow{QP}$) for a general point Q on l

Use a correct method to express PQ^2 (or PQ) in terms of μ

Obtain a correct expression in any form

Carry out a complete method for finding its minimum

Obtain answer 1.22

*OR*2:

Calling (4, 2, 5) A, state $\overrightarrow{PA}$ (or $\overrightarrow{AP}$) in component form, e.g. $\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$

Use a scalar product to find the projection of $\overrightarrow{PA}$ (or $\overrightarrow{AP}$) on l

Obtain correct answer $21/\sqrt{14}$, or equivalent

Use Pythagoras to find the perpendicular

Obtain answer 1.22

*OR*3:

State $\overrightarrow{PA}$ (or $\overrightarrow{AP}$) in component form

Calculate vector product of $\overrightarrow{PA}$ and a direction vector for l

Obtain correct answer, e.g. $4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Divide modulus of the product by that of the direction vector

Obtain answer 1.22 [5]

(ii) *EITHER*

Use scalar product to obtain a relevant equation in a, b and c, e.g. $a + 2b + 3c = 0$

Obtain a second relevant equation, e.g. using $\overrightarrow{PA}$ $a + 4b + 4c = 0$, and solve for one ratio

Obtain $a : b : c = 4 : 1 : -2$, or equivalent

Substitute a relevant point and values of a, b, c in general equation and find d

Obtain correct answer, $4x + y - 2z = 8$, or equivalent

*OR*1: Attempt to calculate vector product of relevant vectors, e.g. $(\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

Obtain two correct components

Obtain correct answer, e.g. $4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Substitute a relevant point and find d

Obtain correct answer, $4x + y - 2z = 8$, or equivalent

*OR*2: Using a relevant point and relevant vectors form a 2-parameter equation for the plane

State a correct equation, e.g.

$\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

State three correct equations in x, y, z, λ and μ

Eliminate λ and μ

Obtain correct answer $4x + y - 2z = 8$, or equivalent [5]

3. M/J 18/P33/Q10

(i) Carry out a correct method for finding a vector equation for AB

Obtain $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{k})$, or equivalent

Equate pair(s) of components AB and l and solve for λ or μ

Obtain correct answer for λ or μ [5]

Verify that all three component equations are not satisfied

(ii) State or imply a direction vector for AP has components $(2+t, 5+2t, -3-2t)$

State or imply that cos 120° equals the scalar product of $\overrightarrow{AP}$ and $\overrightarrow{AB}$ divided by the product of their moduli

Carry out the correct processes for finding the scalar product and the product of the moduli in terms of t, and obtain an equation in terms of t

Obtain the given equation correctly

Solve the quadratic and use a root to find a position vector for P

Obtain position vector $2\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ from $t=-2$, having rejected the root $t=-\frac{2}{3}$ **[6]**

4. O/N 17/P32/Q10

(i) State or imply a correct normal vector to either plane, e.g. $\mathbf{i}+\mathbf{j}+3\mathbf{k}$ or $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$

Carry out correct process for evaluating the scalar product of two normal vectors

Using the correct process for the moduli, divide the scalar product of the two normals by the product of their moduli and evaluate the inverse cosine of the result

Obtain final answer 72.5° or 1.26 radians **[4]**

(ii) *EITHER*: Substitute $y=2$ in both plane equations and solve for x or for z

Obtain $x=3$ and $z=1$

OR: Find the equation of the line of intersection of the planes

Substitute $y=2$ in line equation and solve for

Obtain x = 3 and z = 1

EITHER: Use scalar product to obtain an equation in a, b and c, e.g. $a+b+3c=0$

Form a second relevant equation, e.g. $2a-2b+c=0$, and solve for one ratio, e.g. $a:b$

Obtain final answer $a:b:c=7:5:-4$

Use coordinates of A and values of a, b and c in general equation and find d

Obtain answer $7x+5y-4z=27$, or equivalent

OR1: Calculate the vector product of relevant vectors, e.g. $(\mathbf{i}+\mathbf{j}+3\mathbf{k})\times(2\mathbf{i}-2\mathbf{j}+\mathbf{k})$

Obtain two correct components

Obtain correct answer, e.g. $7\mathbf{i}+5\mathbf{j}-4\mathbf{k}$

Substitute coordinates of A in plane equation with their normal and find d

Obtain answer $7x+5y-4z=27$, or equivalent

OR2: Using relevant vectors, form a two-parameter equation for the plane

State a correct equation, e.g. $\mathbf{r}=3\mathbf{i}+2\mathbf{j}+\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}+3\mathbf{k})+\mu(2\mathbf{i}-2\mathbf{j}+\mathbf{k})$

State 3 correct equations in x, y, z, λ and μ

Eliminate λ and μ

Obtain answer $7x+5y-4z=27$, or equivalent

OR3: Use the direction vector of the line of intersection of the two planes as normal vector to the plane

Two correct components

Three correct components

Substitute coordinates of A in plane equation with their normal and find d

Obtain answer $7x+5y-4z=27$, or equivalent **[7]**

5. O/N 17/P31/Q10, O/N 17/P33/Q10

(i) Equate at least two pairs of components of general points on l and m and solve for λ or for μ

Obtain correct answer for λ or μ, e.g. $\lambda=3$ or $\mu=-2$, $\lambda=0$ or $\mu=-\frac{1}{2}$; or $\lambda=\frac{3}{2}$ or $\mu=-\frac{7}{2}$

Verify that not all three pairs of equations are satisfied and that the lines fail to intersect **[3]**

(ii) Carry out correct process for evaluating scalar product of direction vectors for l and m
Using the correct process for the moduli, divide the scalar product by the product of the moduli and evaluate the inverse cosine of the result
Obtain answer 45° or $\frac{1}{4}\pi$ (0.785) radians [3]

(iii) *EITHER:* Use scalar product to obtain a relevant equation in a, b and c, e.g. $-a+b+4c=0$
Obtain a second equation, e.g. $2a+b-2c=0$ and solve for one ratio, e.g. $a:b$
Obtain $a:b:c=2:-2:1$, or equivalent
Substitute (3, –2, –1) and values of a, b and c in general equation and find d
Obtain answer $2x-2y+z=9$, or equivalent

*OR*1: Attempt to calculate vector product of relevant vectors, e.g $(-\mathbf{i}+\mathbf{j}+4\mathbf{k})\times(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$
Obtain two correct components
Obtain correct answer, e.g. $-6\mathbf{i}+6\mathbf{j}-3\mathbf{k}$
Substitute (3, –2, –1) in $-6x+6y-3z=d$, or equivalent, and find d
Obtain answer $-2x+2y-z=-9$, or equivalent

*OR*2: Using the relevant point and relevant vectors, form a 2-parameter equation for the plane
State a correct equation, e.g. $\mathbf{r}=3\mathbf{i}-2\mathbf{j}-\mathbf{k}+\lambda(-\mathbf{i}+\mathbf{j}+4\mathbf{k})+\mu(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$
State three correct equations in x, y, z, λ and μ
Eliminate λ and μ
Obtain answer $2x-2y+z=9$, or equivalent

*OR*3: Using the relevant point and relevant vectors, form a determinant equation for the plane
State a correct equation, e.g. $\begin{vmatrix} x-3 & y+2 & z+1 \\ -1 & 1 & 4 \\ 2 & 1 & -2 \end{vmatrix}=0$
Attempt to expand the determinant
Obtain two correct cofactors
Obtain answer $-2x+2y-z=-9$, or equivalent [5]

6. M/J 17/P32/Q9

(i) *EITHER:*
Find $\overrightarrow{AP}$ for a general point P on l with parameter λ, e.g.$(8+3\lambda, -3-\lambda, 4+2\lambda)$
Equate scalar product of AP and direction vector of l to zero and solve for λ
Obtain $\lambda=-\frac{5}{2}$ and foot of perpendicular $\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}+3\mathbf{k}$
Carry out a complete method for finding the position vector of the reflection of A in l [5]
Obtain answer $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$

OR:
Find $\overrightarrow{AP}$ for a general point P on l with parameter λ, e.g.$(8+3\lambda, -3-\lambda, 4+2\lambda)$
Differentiate $|AP|^2$ and solve for λ at minimum
Obtain $\lambda=-\frac{5}{2}$ and foot of perpendicular $\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}+3\mathbf{k}$
Carry out a complete method for finding the position vector of the reflection of A in l [5]
Obtain answer $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$

(ii) *EITHER:*
Use scalar product to obtain an equation in a, b and c, e.g. $3a - b + 2c = 0$
Form a second relevant equation, e.g. $9a - b + 8c = 0$ and solve for one ratio, e.g. $a : b$
Obtain final answer $a : b : c = 1 : 1 : -1$ and state plane equation $x + y - z = 0$
*OR*1:
Attempt to calculate vector product of two relevant vectors, e.g. $(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (9\mathbf{i} - \mathbf{j} + 8\mathbf{k})$
Obtain two correct components
Obtain correct answer, e.g. $-6\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$, and state plane equation $-x - y + z = 0$
*OR*2:
Using a relevant point and relevant vectors, attempt to form a 2-parameter equation for the plane, e.g. $\mathbf{r} = 6\mathbf{i} + 6\mathbf{k} + s(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + t(9\mathbf{i} - \mathbf{j} + 8\mathbf{k})$

State 3 correct equations in x, y, z, s and t
Eliminate s and t and state plane equation $x + y - z = 0$, or equivalent
*OR*3:
Using a relevant point and relevant vectors, attempt to form a determinant equation for the

plane, e.g. $\begin{vmatrix} x-3 & y-1 & z-4 \\ 3 & -1 & 2 \\ 9 & -1 & 8 \end{vmatrix} = 0$

Expand a correct determinant and obtain two correct cofactors
Obtain answer $-6x - 6y + 6z = 0$, or equivalent **[3]**

(iii) *EITHER*:
Using the correct processes, divide the scalar product of $\overrightarrow{OA}$ and a normal to the plane by the modulus of the normal or make a recognisable attempt to apply the perpendicular formula

Obtain a correct expression in any form, e.g. $\dfrac{1+2-4}{\sqrt{(1^2+1^2+(-1)^2)}}$, or equivalent

Obtain answer $1/\sqrt{3}$, or exact equivalent
OR1:
Obtain equation of the parallel plane through A, e.g. $x + y - z = -1$
[The f.t. is on the plane found in part (ii).]
Use correct method to find its distance from the origin
Obtain answer $1/\sqrt{3}$, or exact equivalent
OR2:
Form equation for the intersection of the perpendicular through A and the plane
[FT on their **n**]
Solve for λ
$|\lambda \mathbf{n}| = \dfrac{1}{\sqrt{3}}$ **[3]**

7. M/J 17/P31/Q6

(i) State or obtain coordinates (1, 2, 1) for the mid-point of AB
Verify that the midpoint lies on m
State or imply a correct normal vector to the plane, e.g. $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

State or imply a direction vector for the segment AB, e.g. $-4\mathbf{i}-4\mathbf{j}+2\mathbf{k}$

Confirm that m is perpendicular to AB [5]

(ii) State or imply that the perpendicular distance of m from the origin is $\frac{5}{3}$, or unsimplified equivalent

State or imply that n has an equation of the form $2x+2y-z=k$

Obtain answer $2x+2y-z=2$ [3]

8. M/J 17/P33/Q10

(i) Carry out a correct method for finding a vector equation for AB

Obtain $\mathbf{r}=\mathbf{i}-2\mathbf{j}+2\mathbf{k}+\lambda(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$, or equivalent

Equate two pairs of components of general points on AB and l and solve for λ or for μ

Obtain correct answer for λ or μ, e.g. $\lambda=\frac{5}{7}$ or $\mu=\frac{3}{7}$ [5]

Obtain $m=3$

(ii) *EITHER:*

Use scalar product to obtain an equation in a, b and c, e.g. $a-2b-4c=0$

Form a second relevant equation, e.g. $2a+3b-c=0$ and solve for one ratio, e.g. $a : b$

Obtain final answer $a : b : c = 14 : -7 : 7$

Use coordinates of a relevant point and values of a, b and c and find d

Obtain answer $14x-7y+7z=42$, or equivalent

OR 1:

Attempt to calculate the vector product of relevant vectors, e.g.

$(\mathbf{i}-2\mathbf{j}-4\mathbf{k})\times(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$

Obtain two correct components

Obtain correct answer, e.g. $14\mathbf{i}-7\mathbf{j}+7\mathbf{k}$

Substitute coordinates of a relevant point in $14x-7y+7z=d$, or equivalent, and find d

Obtain answer $14x-7y+7z=42$, or equivalent

OR 2:

Using a relevant point and relevant vectors, form a 2–parameter equation for the plane

State a correct equation, e.g. $\mathbf{r}=\mathbf{i}-2\mathbf{j}+2\mathbf{k}+s(\mathbf{i}-2\mathbf{j}-4\mathbf{k})+t(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$

State 3 correct equations in x, y, z, s and t

Eliminate s and t

Obtain answer $2x-y+z=6$, or equivalent

OR 3:

Using a relevant point and relevant vectors, form a determinant equation for the plane

State a correct equation, e.g. $\begin{vmatrix} x-1 & y+2 & z-1 \\ 1 & -2 & -4 \\ 2 & 3 & -1 \end{vmatrix}=0$

Attempt to expand the determinant
Obtain or imply two correct cofactors
Obtain answer $14x-7y+7z=42$, or equivalent [5]

9. O/N 16/P32/Q8, O/N 16/P31/Q8

(i) State or imply a correct normal vector to either plane, e.g. $3\mathbf{i}+\mathbf{j}-\mathbf{k}$ or $\mathbf{i}-\mathbf{j}+2\mathbf{k}$
Use correct method to calculate their scalar product
Show value is zero and planes are perpendicular [3]

(ii) *EITHER*: Carry out a complete strategy for finding a point on l the line of intersection
Obtain such a point, e.g. (0, 7, 5), (1, 0, 1), (5/4, –7/4, 0)
EITHER: State two equations for a direction vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ for l,
e.g. $3a+b-c=0$ and $a-b+2c=0$
Solve for one ratio, e.g. $a : b$
Obtain $a : b : c = 1 : -7 : -4$, or equivalent
State a correct answer, e.g. $\mathbf{r}=7\mathbf{j}+5\mathbf{k}+\lambda(\mathbf{i}-7\mathbf{j}-4\mathbf{k})$

*OR*1: Obtain a second point on l, e.g. (1, 0, 1)
Subtract vectors and obtain a direction vector for l
Obtain $-\mathbf{i}+7\mathbf{j}+4\mathbf{k}$, or equivalent
State a correct answer, e.g. $\mathbf{r}=\mathbf{i}+\mathbf{k}+\lambda(-\mathbf{i}+7\mathbf{j}+4\mathbf{k})$

*OR*2: Attempt to find the vector product of the two normal vectors
Obtain two correct components of the product
Obtain $\mathbf{i}-7\mathbf{j}-4\mathbf{k}$, or equivalent
State a correct answer, e.g. $\mathbf{r}=7\mathbf{j}+5\mathbf{k}+\lambda(\mathbf{i}-7\mathbf{j}-4\mathbf{k})$

*OR*1: Express one variable in terms of a second variable
Obtain a correct simplified expression, e.g. $y=7-7x$
Express the third variable in terms of the second
Obtain a correct simplified expression, e.g. $z=5-4x$
Form a vector equation for the line
Obtain a correct equation, e.g. $\mathbf{r}=7\mathbf{j}+5\mathbf{k}+\lambda(\mathbf{i}-7\mathbf{j}-4\mathbf{k})$

*OR*2: Express one variable in terms of a second variable
Obtain a correct simplified expression, e.g. $z=5-4x$
Express the same variable in terms of the third
Obtain a correct simplified expression e.g. $z=(7+4y)/7$
Form a vector equation for the line
Obtain a correct equation, e.g. $\mathbf{r}=\frac{5}{4}\mathbf{i}-\frac{7}{4}\mathbf{j}+\lambda(-\frac{1}{4}\mathbf{i}+\frac{7}{4}\mathbf{j}+\mathbf{k})$ [6]

10. O/N 16/P33/Q10

(i) Express general point of l in component form e.g. $(1+2\lambda,\ 2-\lambda,\ 1+\lambda)$
Using the correct process for the modulus form an equation in λ
Reduce the equation to a quadratic, e.g. $6\lambda^2+2\lambda-4=0$
Solve for λ (usual requirements for solution of a quadratic)
Obtain final answers $-\mathbf{i}+3\mathbf{j}$ and $\frac{7}{3}\mathbf{i}+\frac{4}{3}\mathbf{j}+\frac{5}{3}\mathbf{k}$ [5]

(ii) Using the correct process, find the scalar product of a direction vector for l and a normal for p
Using the correct process for the moduli, divide the scalar product by the product of the moduli and equate the result to $\frac{2}{3}$

State a correct equation in any form, e.g. $\dfrac{2a-1+1}{\sqrt{(a^2+1+1)}.\sqrt{(2^2+(-1)^2+1)}}=\pm\dfrac{2}{3}$

Solve for a^2

Obtain answer $a=\pm 2$ [5]

11. M/J 16/P32/Q9

(i) *Either* state or imply $\overrightarrow{AB}$ or $\overrightarrow{BC}$ in component form, *or* state position vector of midpoint of $\overrightarrow{AC}$

Use a correct method for finding the position vector of D

Obtain answer $3\mathbf{i}+3\mathbf{j}+\mathbf{k}$, or equivalent

EITHER: Using the correct process for the moduli, compare lengths of a pair of adjacent sides,

e.g. AB and BC

Show that $ABCD$ has a pair of adjacent sides that are equal

OR: Calculate scalar product $\overrightarrow{AC}.\overrightarrow{BD}$ or equivalent

Show that $ABCD$ has perpendicular diagonals

(ii) *EITHER*: State $a+2b+3c=0$ or $2a+b-2c=0$

Obtain two relevant equations and solve for one ratio, e.g. $a:b$

Obtain $a:b:c=-7:8:-3$, or equivalent

Substitute coordinates of a relevant point in $-7x+8y-3z=d$, and evaluate

Obtain answer $-7x+8y-3z=29$, or equivalent

*OR*1:Attempt to calculate vector product of relevant vectors,

e.g. $(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\times(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$

Obtain two correct components of the product

Obtain correct product, e.g. $-7\mathbf{i}+8\mathbf{j}-3\mathbf{k}$

Substitute coordinates of a relevant point in $-7x+8y-3z=d$ and evaluate d

Obtain answer $-7x+8y-3z=29$ or equivalent

*OR*2:Attempt to form a 2-parameter equation with relevant vectors

State a correct equation, e.g. $\mathbf{r}=2\mathbf{i}+5\mathbf{j}-\mathbf{k}+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})+\mu(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$

State 3 equations in x, y, z, λ and μ

Eliminate λ and μ

Obtain answer $-7x+8y-3z=29$, or equivalent

*OR*3:Using a relevant point and relevant direction vectors, form a determinant equation for the plane

State a correct equation, e.g. $\begin{vmatrix} x-2 & y-5 & z+1 \\ 1 & 2 & 3 \\ 2 & 1 & -2 \end{vmatrix}=0$

Attempt to expand the determinant

Obtain correct values of two cofactors

Obtain answer $-7x+8y-3z=29$, or equivalent

12. M/J 16/P31/Q9

(i) *EITHER*: Obtain a vector parallel to the plane, e.g. $\overrightarrow{AB}=\mathbf{i}-2\mathbf{j}-3\mathbf{k}$

Use scalar product to obtain an equation in a, b, c e.g. $a-2b-3c=0$, $a+b-c=0$, or $3b+2c=0$

State two correct equations

Solve to obtain ratio $a:b:c$

Obtain $a:b:c=5:-2:3$

Obtain equation $5x-2y+3z=5$, or equivalent

*OR*1: Substitute for two points, e.g. *A* and *B*, and obtain $a+3b+2c=d$ and $2a+b-c=d$

Substitute for another point, e.g. *C*, to obtain a third equation and eliminate one unknown entirely from all three equations

Obtain two correct equations in three unknowns, e.g. in *a, b, c*

Solve to obtain their ratio

Obtain $a:b:c=5:-2:3$, $a:c:d=5:3:5$, $a:b:d=5:-2:5$, or $b:c:d=-2:3:5$

Obtain equation $5x-2y+3z=5$, or equivalent

*OR*2: Obtain a vector parallel to the plane, e.g. $\overrightarrow{AC}=\mathbf{i}+\mathbf{j}-\mathbf{k}$

Obtain a second such vector and calculate their vector product, e.g. $(\mathbf{i}-2\mathbf{j}-3\mathbf{k})\times(\mathbf{i}+\mathbf{j}-\mathbf{k})$

Obtain two correct components of the product

Obtain correct answer e.g. $5\mathbf{i}-2\mathbf{j}+3\mathbf{k}$

Substitute in $5x-2y+3z=d$ to find d

Obtain equation $5x-2y+3z=5$, or equivalent

*OR*3: Obtain a vector parallel to the plane, e.g. $\overrightarrow{BC}=3\mathbf{j}+2\mathbf{k}$

Obtain a second such vector and form correctly a 2-parameter equation for the plane

Obtain a correct equation, e.g. $\mathbf{r}=\mathbf{i}+3\mathbf{j}+2\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j}-3\mathbf{k})+\mu(3\mathbf{j}+2\mathbf{k})$

State three correct equations in x, y, z, λ, μ

Eliminate λ and μ

Obtain equation $3x-2y+3z=5$, or equivalent

(ii) Correctly form an equation for the line through *D* parallel to *OA*

Obtain a correct equation e.g. $\mathbf{r}=-3\mathbf{i}+\mathbf{j}+2\mathbf{k}+\lambda(\mathbf{i}+3\mathbf{j}+2\mathbf{k})$

Substitute components in the equation of the plane and solve for λ

Obtain $\lambda=2$ and position vector $-\mathbf{i}+7\mathbf{j}+6\mathbf{k}$ for *P*

Obtain the given answer correctly

13. M/J 16/P33/Q8

(i) State a correct equation for *AB* in any form, e.g. $\mathbf{r}=\mathbf{i}+\mathbf{j}+\mathbf{k}+\lambda(\mathbf{i}-\mathbf{j}+2\mathbf{k})$, or equivalent

Equate at least two pairs of components of *AB* and *l* and solve for λ or for μ

Obtain correct answer for λ or for μ, e.g. $\lambda=-1$ or $\mu=2$

Show that not all three equations are not satisfied and that the lines do not intersect

(ii) *EITHER*: Find $\overrightarrow{AP}$ (or $\overrightarrow{PA}$) for a general point *P* on *l*, e.g. $(1-\mu)\mathbf{i}+(-3+2\mu)\mathbf{j}+(-2+\mu)\mathbf{k}$

Calculate the scalar product of $\overrightarrow{AP}$ and a direction vector for *l* and equate to zero

Solve and obtain $\mu=\frac{3}{2}$

Carry out a method to calculate *AP* when $\mu=\frac{3}{2}$

Obtain the given answer $\frac{1}{\sqrt{2}}$ correctly

OR 1:Find $\overrightarrow{AP}$ (or $\overrightarrow{PA}$) for a general point *P* on *l*

Use correct method to express AP^2 (or *AP*) in terms of μ

Obtain a correct expression in any form, e.g. $(1-\mu)^2+(-3+2\mu)^2+(-2+\mu)^2$

Carry out a complete method for finding its minimum

Obtain the given answer correctly

OR 2:Calling (2, −2, −1) *C*, state $\overrightarrow{AC}$ (or $\overrightarrow{CA}$) in component form, e.g. $\mathbf{i}-3\mathbf{j}-2\mathbf{k}$

Use a scalar product to find the projection of $\overrightarrow{AC}$ (or $\overrightarrow{CA}$) on *l*

Obtain correct answer in any form, e.g. $\frac{9}{\sqrt{6}}$

Use Pythagoras to find the perpendicular
Obtain the given answer correctly
OR 3:State $\overrightarrow{AC}$(or $\overrightarrow{CA}$) in component form
Calculate vector product of $\overrightarrow{AC}$ and a direction vector for l, e.g. $(\mathbf{i}-3\mathbf{j}-2\mathbf{k})\times(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$
Obtain correct answer in any form, e.g. $\mathbf{i}+\mathbf{j}-\mathbf{k}$
Divide modulus of the product by that of the direction vector
Obtain the given answer correctly

14. O/N 15/P32/Q7, O/N 15/P31/Q7

(i) Use correct method to form a vector equation for AB
Obtain a correct equation, e.g. $\mathbf{r}=\mathbf{i}+2\mathbf{j}+\lambda(2\mathbf{i}-2\mathbf{j}+\mathbf{k})$ or $\mathbf{r}=3\mathbf{i}+\mathbf{k}+\mu(2\mathbf{i}-2\mathbf{j}+\mathbf{k})$

(ii) Using a direction vector for AB and a relevant point, obtain an equation for m in any form
Obtain answer $2x-2y+z=4$, or equivalent

(iii) Express general point of AB in component form, e.g. $(1+2\lambda, 2-2\lambda, \lambda)$ or $(3+2\mu, -2\mu, 1+\mu)$
Substitute in equation of m and solve for λ or for μ
Obtain final answer $\frac{7}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$ for the position vector of N, from $\lambda=\frac{2}{3}$ or $\mu=-\frac{1}{3}$
Carry out a correct method for finding CN
Obtain the given answer $\sqrt{13}$
[The f.t. is on the direction vector for AB.]

15. O/N 15/P33/Q8

(i) Express a general point on the line in single component form, e.g. $(\lambda, 2-3\lambda, -8+4\lambda)$, substitute in equation of plane and solve for λ
Obtain $\lambda=3$
Obtain $(3,-7,4)$

(ii) State or imply normal vector to plane is $4\mathbf{i}-\mathbf{j}+5\mathbf{k}$
Carry out process for evaluating scalar product of two relevant vectors
Using the correct process for the moduli, divide the scalar product by the product of the moduli and evaluate $\sin^{-1}$ or $\cos^{-1}$ of the result.
Obtain $54.8°$ or 0.956 radians

(iii) Either Find at least one position of C by translating by appropriate multiple of direction vector $\mathbf{i}-3\mathbf{j}+4\mathbf{k}$ from A or B
Obtain $(-3,11,-20)$
Obtain $(9,-25,28)$

Or Form quadratic equation in λ by considering $BC^2=4AB^2$
Obtain $26\lambda^2-156\lambda-702=0$ or equivalent and hence $\lambda=-3, \lambda=9$
Obtain $(-3,11,-20)$ and $(9,-25,28)$

16. M/J 15/P32/Q10

(i) Carry out a correct method for finding a vector equation for AB
Obtain $\mathbf{r}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})$, or equivalent
Equate at least two pairs of components of general points on AB and l and solve for λ or for μ
Obtain correct answer for λ or μ, e.g. $\lambda=1$ or $\mu=0$; $\lambda=-\frac{4}{5}$ or $\mu=\frac{3}{5}$;
or $\lambda=\frac{1}{4}$ or $\mu=-\frac{3}{2}$
Verify that not all three pairs of equations are satisfied and that the lines fail to intersect

(ii) *EITHER*
Obtain a vector parallel to the plane and not parallel to l, e.g. $\mathbf{i}-2\mathbf{j}+\mathbf{k}$
Use scalar product to obtain an equation in a, b and c, e.g. $3a+b-c=0$
Form a second relevant equation, e.g. $a-2b+c=0$ and solve for one ratio,
e.g. $a:b$
Obtain final answer $a:b:c=1:4:7$ A1
Use coordinates of a relevant point and values of a, b and c in general equation and find d
Obtain answer $x+4y+7z=19$, or equivalent

*OR*1: Obtain a vector parallel to the plane and not parallel to l, e.g. $\mathbf{i}-2\mathbf{j}+\mathbf{k}$
Obtain a second relevant vector parallel to the plane and attempt to calculate their vector product, e.g. $(\mathbf{i}-2\mathbf{j}+\mathbf{k})\times(3\mathbf{i}+\mathbf{j}-\mathbf{k})$
Obtain two correct components
Obtain correct answer, e.g. $\mathbf{i}+4\mathbf{j}+7\mathbf{k}$
Substitute coordinates of a relevant point in $x+4y+7z=d$, or equivalent, and find d
Obtain answer $x+4y+7z=19$, or equivalent

*OR*2: Obtain a vector parallel to the plane and not parallel to l, e.g. $\mathbf{i}-2\mathbf{j}+\mathbf{k}$
Using a relevant point and second relevant vector, form a 2-parameter equation for the plane
State a correct equation, e.g. $\mathbf{r}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}+s(\mathbf{i}-2\mathbf{j}+\mathbf{k})+t(3\mathbf{i}+\mathbf{j}-\mathbf{k})$
State 3 correct equations in x, y, z, s and t
Eliminate s and t
Obtain answer $x+4y+7z=19$, or equivalent

*OR*3: Using the coordinates of A and two points on l, state three simultaneous equations in a, b, c and d, e.g. $a+b+2c=d$, $2a-b+3c=d$ and $4a+2b+c=d$
Solve and find one ratio, e.g. $a:b$
State one correct ratio
Obtain a correct ratio of three of the unknowns, e.g. $a:b:c=1:4:7$, or equivalent
Either use coordinates of a relevant point and the found ratio to find the fourth unknown, e.g. d, or find the ratio $a:b:c:d$
Obtain answer $x+4y+7z=19$, or equivalent

*OR*4: Obtain a vector parallel to the plane and not parallel to l, e.g. $\mathbf{i}-2\mathbf{j}+\mathbf{k}$
Using a relevant point and second relevant vector, form a determinant equation for the plane

State a correct equation, e.g. $\begin{vmatrix} x-2 & y+1 & z-3 \\ 1 & -2 & 1 \\ 3 & 1 & -1 \end{vmatrix}=0$

Attempt to expand the determinant
Obtain or imply two correct cofactors
Obtain answer $x+4y+7z=19$, or equivalent **[6]**

17. M/J 15/P31/Q6

(i) Obtain $\pm\begin{pmatrix}2\\-3\\-4\end{pmatrix}$ as direction vector of l_1

State that two direction vectors are not parallel
Express general point of l_1 or l_2 in component form, e.g. $(2\lambda, 1-3\lambda, 5-4\lambda)$
or $(7+\mu, l+2\mu, 1+5\mu)$

Equate at least two pairs of components and solve for λ or for μ
Obtain correct answers for λ and μ
Verify that all three component equations are not satisfied (with no errors seen) [6]

(ii) Carry out correct process for evaluating scalar product of $\begin{pmatrix}1\\2\\5\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\end{pmatrix}$

Use correct process for finding modulus and evaluating inverse cosine
Obtain $79.5°$ or 1.39 radians [3]

18. M/J 15/P33/Q9

(i) State or imply a correct normal vector to either plane, e.g. $\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, or $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$
Carry out correct process for evaluating the scalar product of two normal vectors
Using the correct process for the moduli, divide the scalar product of the two normals by the product of their moduli and evaluate the inverse cosine of the result
Obtain answer $85.9°$ or 1.50 radians 4

(ii) *EITHER*: Carry out a complete strategy for finding a point on l
Obtain such a point, e.g. $(0, 2, 1)$

EITHER: State two equations for a direction vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ for l,
e.g. $a+3b-2c=0$
and $2a+b+3c=0$
Solve for one ratio, e.g. $a:b$
Obtain $a:b:c=11:-7:-5$
State a correct answer, e.g. $\mathbf{r}=2\mathbf{j}+\mathbf{k}+\lambda(11\mathbf{i}-7\mathbf{j}-5\mathbf{k})$

*OR*1: Obtain a second point on l, e.g. $\left(\frac{22}{7}, 0, -\frac{3}{7}\right)$
Subtract position vectors and obtain a direction vector for l
Obtain $22\mathbf{i}-14\mathbf{j}-10\mathbf{k}$, or equivalent
State a correct answer, e.g. $\mathbf{r}=2\mathbf{j}+\mathbf{k}+\lambda(22\mathbf{i}-14\mathbf{j}-10\mathbf{k})$

*OR*2: Attempt to find the vector product of the two normal vectors
Obtain two correct components
Obtain $11\mathbf{i}-7\mathbf{j}-5\mathbf{k}$, or equivalent
State a correct answer, e.g. $\mathbf{r}=2\mathbf{j}+\mathbf{k}+\lambda(11\mathbf{i}-7\mathbf{j}-5\mathbf{k})$

*OR*3: Express one variable in terms of a second
Obtain a correct simplified expression, e.g. $x=(22-11y)/7$
Express the same variable in terms of the third
Obtain a correct simplified expression, e.g. $x=(11-11z)/5$
Form a vector equation for the line M1
State a correct answer, e.g. $\mathbf{r}=2\mathbf{j}+\mathbf{k}+\lambda\left(\mathbf{i}-\frac{7}{11}\mathbf{j}-\frac{5}{11}\mathbf{k}\right)$

*OR*4: Express one variable in terms of a second
Obtain a correct simplified expression, e.g. $y=(22-7x)/11$
Express the third variable in terms of the second
Obtain a correct simplified expression, e.g. $z=(11-5x)/11$
Form a vector equation for the line
State a correct answer, e.g. $\mathbf{r}=2\mathbf{j}+\mathbf{k}+\lambda\left(\mathbf{i}-\frac{7}{11}\mathbf{j}-\frac{5}{11}\mathbf{k}\right)$ 6

[The √ marks are dependent on all M marks being earned.]

19. O/N 14/P32/Q10, O/N 14/P31/Q10

(i) *EITHER:* Find $\overrightarrow{AP}$ (or $\overrightarrow{PA}$) for a point P on l with parameter λ, e.g. $\mathbf{i} - 17\mathbf{j} + 4\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

Calculate scalar product of AP and a direction vector for l and equate to zero
Solve and obtain $\lambda = 3$
Carry out a complete method for finding the length of AP
Obtain the given answer 15 correctly

OR1: Calling $(4, -9, 9)$ B, state $\overrightarrow{BA}$ (or $\overrightarrow{AB}$) in component form, e.g. $-\mathbf{i} + 17\mathbf{j} - 4\mathbf{k}$
Calculate vector product of $\overrightarrow{BA}$ and a direction vector for l, e.g. $(-\mathbf{i} + 17\mathbf{j} - 4\mathbf{k}) \times (-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$
Obtain correct answer, e.g. $-30\mathbf{i} + 6\mathbf{j} + 33\mathbf{k}$
Divide the modulus of the product by that of the direction vector
Obtain the given answer correctly

OR2: State $\overrightarrow{BA}$ (or $\overrightarrow{AB}$) in component form
Use a scalar product to find the projection of BA (or AB) on l
Obtain correct answer in any form, e.g. $\dfrac{27}{\sqrt{9}}$
Use Pythagoras to find the perpendicular
Obtain the given answer correctly

OR3: State $\overrightarrow{BA}$ (or $\overrightarrow{AB}$) in component form
Use a scalar product to find the cosine of ABP
Obtain correct answer in any form, e.g. $\dfrac{27}{\sqrt{9}.\sqrt{306}}$
Use trig. to find the perpendicular
Obtain the given answer correctly

OR4: State $\overrightarrow{BA}$ (or $\overrightarrow{AB}$) in component form
Find a second point C on l and use the cosine rule in triangle ABC to find the cosine of angle A, B, or C, or use a vector product to find the area of ABC
Obtain correct answer in any form
Use trig. or area formula to find the perpendicular
Obtain the given answer correctly

OR5: State correct $\overrightarrow{AP}$ (or $\overrightarrow{PA}$) for a point P on l with parameter λ in any form
Use correct method to express AP^2 (or AP) in terms of λ
Obtain a correct expression in any form, e.g. $(1-2\lambda)^2 + (-17+\lambda)^2 + (4-2\lambda)^2$
Carry out a method for finding its minimum (using calculus, algebra or Pythagoras)
Obtain the given answer correctly **[5]**

(ii) *EITHER:* Substitute coordinates of a general point of l in equation of plane and either equate constant terms or equate the coefficient of λ to zero, obtaining an equation in a and b
Obtain a correct equation, e.g. $4a - 9b - 27 + 1 = 0$
Obtain a second correct equation, e.g. $-2a + b + 6 = 0$
Solve for a or for b
Obtain $a = 2$ and $b = -2$

OR: Substitute coordinates of a point of l and obtain a correct equation, e.g. $4a - 9b = 26$

EITHER: Find a second point on l and obtain an equation in a and b
Obtain a correct equation

OR: Calculate scalar product of a direction vector for l and a vector normal to the plane and equate to zero
Obtain a correct equation, e.g. $-2a + b + 6 = 0$

Solve for a or for b
Obtain $a = 2$ and $b = -2$ [5]

20. O/N 14/P33/Q7

(i) State at least two of the equations $1+\lambda = a+\mu$, $4 = 2+2\mu$, $-2+3\lambda = -2+3a\mu$
Solve for λ or for μ
Obtain $\lambda = a$ (or $\lambda = a+\mu-1$) and $\mu = 1$
Confirm values satisfy third equation [4]

(ii) State or imply point of intersection is $(a+1, 4, 3a-2)$
Use correct method for the modulus of the position vector and equate to 9, following their point of intersection
Solve a three-term quadratic equation in a $\left(a^2 - a - 6 = 0\right)$
Obtain –2 and 3 [4]

21. M/J 14/P32/Q10

(i) *EITHER:* State or imply $\overrightarrow{AB}$ and $\overrightarrow{AC}$ correctly in component form
Using the correct processes evaluate the scalar product $\overrightarrow{AB}.\overrightarrow{AC}$, or equivalent
Using the correct process for the moduli divide the scalar product by the product of the moduli
Obtain answer $\frac{20}{21}$

OR: Use correct method to find lengths of all sides of triangle ABC
Apply cosine rule correctly to find the cosine of angle BAC
Obtain answer $\frac{20}{21}$ 4

(ii) State an exact value for the sine of angle BAC, e.g. $\sqrt{41}/21$
Use correct area formula to find the area of triangle ABC
Obtain answer $\frac{1}{2}\sqrt{41}$, or exact equivalent 3

[SR: Allow use of a vector product, e.g. $\overrightarrow{AB}\times\overrightarrow{AC} = -6\mathbf{i}+2\mathbf{j}-\mathbf{k}$ B1√. Using correct process for the modulus, divide the modulus by 2 M1. Obtain answer $\frac{1}{2}\sqrt{41}$ A1.]

(iii) *EITHER*: State or obtain $b = 0$
Equate scalar product of normal vector and $\overrightarrow{BC}$ (or $\overrightarrow{CB}$) to zero
Obtain $a + b - 4c = 0$ (or $a - 4c = 0$)
Substitute a relevant point in $4x + z = d$ and evaluate d
Obtain answer $4x + z = 9$, or equivalent

*OR*1: Attempt to calculate vector product of relevant vectors, e.g. $(\mathbf{j})\times(\mathbf{i}+\mathbf{j}-4\mathbf{k})$
Obtain two correct components of the product
Obtain correct product, e.g. $-4\mathbf{i} - \mathbf{k}$
Substitute a relevant point in $4x + z = d$ and evaluate d
Obtain $4x + z = 9$, or equivalent

*OR*2: Attempt to form 2-parameter equation for the plane with relevant vectors
State a correct equation, e.g. $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(\mathbf{j}) + \mu(\mathbf{i}+\mathbf{j}-4\mathbf{k})$

State 3 equations in x, y, z, λ and μ
Eliminate μ
Obtain answer $4x + z = 9$, or equivalent
State or obtain $b = 0$

OR3: Substitute for B and C in the plane equation and obtain $2a + c = d$ and $3a - 3c = d$ (or $2a + 4b + c = d$ and $3a + 5b - 3c = d$)
Solve for one ratio, e.g. $a : d$
Obtain $a : c : d$, or equivalent
Obtain answer $4x + z = 9$, or equivalent

OR4: Attempt to form a determinant equation for the plane with relevant vectors

State a correct equation, e.g. $\begin{vmatrix} x-2 & y-4 & z-1 \\ 0 & 1 & 0 \\ 1 & 1 & -4 \end{vmatrix} = 0$

Attempt to use a correct method to expand the determinant
Obtain two correct terms of a 3-term expansion, or equivalent
Obtain answer $4x + z = 9$, or equivalent **5**

22. M/J 14/P31/Q7

(i) Obtain $2x - 3y + 6z$ for LHS of equation [2]
Obtain $2x - 3y + 6z = 23$

(ii) Either Use correct formula to find perpendicular distance

Obtain unsimplified value $\dfrac{\pm 23}{\sqrt{2^2 + (-3)^2 + 6^2}}$, following answer to (i)

Obtain $\dfrac{23}{7}$ or equivalent [3]

OR 1 Use scalar product of $(4, -1, 2)$ and a vector normal to the plane

Use unit normal to plane to obtain $\pm\dfrac{(8+3+12)}{\sqrt{49}}$

Obtain $\dfrac{23}{7}$ or equivalent [3]

OR 2 Find parameter intersection of p and $\mathbf{r} = \mu(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$

Obtain $\mu = \dfrac{23}{49}$ [and $\left(\dfrac{46}{49}, -\dfrac{69}{49}, \dfrac{138}{49}\right)$ as foot of perpendicular]

Obtain distance $\dfrac{23}{7}$ or equivalent [3]

(iii) Either Recognise that plane is $2x - 3y + 6z = k$ and attempt use of formula for perpendicular distance to plane at least once

Obtain $\dfrac{|23 - k|}{7} = 14$ or equivalent [3]
Obtain $2x - 3y + 6z = 121$ and $2x - 3y + 6z = -75$

OR Recognise that plane is $2x - 3y + 6z = k$ and attempt to find at least one point on q using l with $\lambda = \pm 2$
Obtain $2x - 3y + 6z = 121$ [3]
Obtain $2x - 3y + 6z = -75$

23. M/J 14/P33/Q10

(i) Express general point of l in component form, e.g. $(1+3\lambda, 2-2\lambda, -1+2\lambda)$
Substitute in given equation of p and solve for λ
Obtain final answer $-\frac{1}{2}\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, or equivalent, from $\lambda=-\frac{1}{2}$ 3

(ii) State or imply a vector normal to the plane, e.g. $2\mathbf{i}+3\mathbf{j}-5\mathbf{k}$
Using the correct process, evaluate the scalar product of a direction vector for l and a normal for p
Using the correct process for the moduli, divide the scalar product by the product of the moduli and find the inverse sine or cosine of the result
Obtain answer 23.2° (or 0.404 radians) 4

(iii) *EITHER*: State $2a+3b-5c=0$ or $3a-2b+2c=0$
Obtain two relevant equations and solve for one ratio, e.g. $a:b$
Obtain $a:b:c=4:19:13$, or equivalent
Substitute coordinates of a relevant point in $4x+19y+13z=d$, and evaluate d
Obtain answer $4x+19y+13z=29$, or equivalent

OR1: Attempt to calculate vector product of relevant vectors, e.g. $(2\mathbf{i}+3\mathbf{j}-5\mathbf{k})\times(3\mathbf{i}-2\mathbf{j}+2\mathbf{k})$
Obtain two correct components of the product
Obtain correct product, e.g. $-4\mathbf{i}-19\mathbf{j}-13\mathbf{k}$
Substitute coordinates of a relevant point in $4x+19y+13z=d$
Obtain answer $4x+19y+13z=29$, or equivalent

OR2: Attempt to form a 2-parameter equation with relevant vectors
State a correct equation, e.g. $\mathbf{r}=\mathbf{i}+2\mathbf{j}-\mathbf{k}+\lambda(2\mathbf{i}+3\mathbf{j}-5\mathbf{k})+\mu(3\mathbf{i}-2\mathbf{j}+2\mathbf{k})$
State 3 equations in x, y, z, λ and μ
Eliminate λ and μ
Obtain answer $4x+19y+13z=29$, or equivalent

OR3: Using a relevant point and relevant direction vectors, form a determinant equation for the plane
State a correct equation, e.g. $\begin{vmatrix} x-1 & y-2 & z+1 \\ 2 & 3 & -5 \\ 3 & -2 & 2 \end{vmatrix}=0$
Attempt to expand the determinant
Obtain correct values of two cofactors
Obtain answer $4x+19y+13z=29$, or equivalent 5

24. O/N 13/P32/Q9

(i) *EITHER*: Obtain a vector parallel to the plane, e.g. $\overrightarrow{AB}=-2\mathbf{i}+4\mathbf{j}-\mathbf{k}$
Use scalar product to obtain an equation in a, b, c, e.g. $-2a+4b-c=0$, $3a-3b+3c=0$, or $a+b+2c=0$
Obtain two correct equations in a, b, c
Solve to obtain ratio $a:b:c$
Obtain $a:b:c=3:1:-2$, or equivalent
Obtain equation $3x+y-2z=1$, or equivalent

OR1: Substitute for two points, e.g. A and B, and obtain $2a-b+2c=d$ and $3b+c=d$
Substitute for another point, e.g. C, to obtain a third equation and eliminate one unknown entirely from the three equations
Obtain two correct equations in three unknowns, e.g. in a, b, c
Solve to obtain their ratio, e.g. $a:b:c$
Obtain $a:b:c=3:1:-2$, $a:c:d=3:-2:1$, $a:b:d=3:1:1$ or $b:c:d=-1:-2:1$
Obtain equation $3x+y-2z=1$, or equivalent

OR2: Obtain a vector parallel to the plane, e.g. $\overrightarrow{BC} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$
Obtain a second such vector and calculate their vector product e.g. $(-2\mathbf{i}+4\mathbf{j}-\mathbf{k})\times(3\mathbf{i}-3\mathbf{j}+3\mathbf{k})$
Obtain two correct components of the product
Obtain correct answer, e.g. $9\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$
Substitute in $9x + 3y - 6z = d$ to find d
Obtain equation $9x + 3y - 6z = 3$, or equivalent

OR3: Obtain a vector parallel to the plane, e.g. $\overrightarrow{AC} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$
Obtain a second such vector and form correctly a 2-parameter equation for the plane
Obtain a correct equation, e.g. $\mathbf{r} = 3\mathbf{i} + 4\mathbf{k} + \lambda(-2\mathbf{i}+4\mathbf{j}-\mathbf{k}) + \mu(\mathbf{i}+\mathbf{j}+2\mathbf{k})$
State three correct equations in x, y, z, λ, μ
Eliminate λ and μ
Obtain equation $3x + y - 2z = 1$, or equivalent **[6]**

(ii) Obtain answer $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, or equivalent **[1]**

(iii) *EITHER:* Use $\dfrac{\overrightarrow{OA}.\overrightarrow{OD}}{|\overrightarrow{OD}|}$ to find projection *ON* of *OA* onto *OD*

Obtain $ON = \dfrac{4}{3}$
Use Pythagoras in triangle *OAN* to find *AN*
Obtain the given answer

OR1: Calculate the vector product of $\overrightarrow{OA}$ and $\overrightarrow{OD}$
Obtain answer $6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$
Divide the modulus of the vector product by the modulus of $\overrightarrow{OD}$
Obtain the given answer

OR2: Taking general point *P* of *OD* to have position vector $\lambda(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$, form an equation in λ by either equating the scalar product of $\overrightarrow{AP}$ and $\overrightarrow{OP}$ to zero, or using Pythagoras in triangle *OPA*, or setting the derivative of $|\overrightarrow{AP}|$ to zero
Solve and obtain $\lambda = \dfrac{4}{9}$
Carry out method to calculate *AP* when $\lambda = \dfrac{4}{9}$
Obtain the given answer

OR3: Use a relevant scalar product to find the cosine of *AOD* or *ADO*
Obtain $\cos AOD = \dfrac{4}{9}$ or $\cos ADO = \dfrac{5}{3\sqrt{10}}$, or equivalent
Use trig to find the length of the perpendicular
Obtain the given answer

OR4: Use cosine formula in triangle *AOD* to find cos *AOD* or cos *ADO*
Obtain $\cos AOD = \dfrac{8}{18}$ or $\cos ADO = \dfrac{10}{6\sqrt{10}}$, or equivalent
Use trig to find the length of the perpendicular
Obtain the given answer **[4]**

25. O/N 13/P33/Q6

(i) Find scalar product of the normals to the planes
Using the correct process for the moduli, divide the scalar product by the product of the moduli and find $\cos^{-1}$ of the result.
Obtain 67.8° (or 1.18 radians) [3]

(ii) EITHER Carry out complete method for finding point on line
Obtain one such point, e.g. $(2,-3,0)$ or $\left(\frac{17}{7},0,\frac{6}{7}\right)$ or $(0,-17,-4)$ or …

Either State $3a-b+2c=0$ and $a+b-4c=0$ or equivalent
Attempt to solve for one ratio, e.g. $a:b$
Obtain $a:b:c=1:7:2$ or equivalent
State a correct final answer, e.g. $\mathbf{r}=[2,-3,0]+\lambda[1,7,2]$

Or 1 Obtain a second point on the line
Subtract position vectors to obtain direction vector
Obtain [1, 7, 2] or equivalent
State a correct final answer, e.g. $\mathbf{r}=[2,-3,0]+\lambda[1,7,2]$

Or 2 Use correct method to calculate vector product of two normals
Obtain two correct components
Obtain [2, 14, 4] or equivalent
State a correct final answer, e.g. $\mathbf{r}=[2,-3,0]+\lambda[1,7,2]$
[√ is dependent on both M marks in all three cases]

OR 3 Express one variable in terms of a second variable
Obtain a correct simplified expression, e.g. $x=\frac{1}{2}(4+z)$
Express the first variable in terms of third variable
Obtain a correct simplified expression, e.g. $x=\frac{1}{7}(17+y)$
Form a vector equation for the line
State a correct final answer, e.g. $\mathbf{r}=[0,-17,-4]+\lambda[1,7,2]$

OR 4 Express one variable in terms of a second variable
Obtain a correct simplified expression, e.g. $z=2x-4$
Express third variable in terms of the second variable
Obtain a correct simplified expression, e.g. $y=7x-17$
Form a vector equation for the line
State a correct final answer, e.g. $\mathbf{r}=[0,-17,-4]+\lambda[1,7,2]$ [6]

26. M/J 13/P32/Q10

(i) Carry out a correct method for finding a vector equation for AB
Obtain $\mathbf{r}=2\mathbf{i}-3\mathbf{j}+2\mathbf{k}+\lambda(3\mathbf{i}+\mathbf{j}-\mathbf{k})$ or
$\mathbf{r}=\mu(2\mathbf{i}+3\mathbf{j}+2\mathbf{k})+(1-\mu)(5\mathbf{i}-2\mathbf{j}+\mathbf{k})$, or equivalent
Substitute components in equation of p and solve for λ or for μ
Obtain $\lambda=\frac{3}{2}$ or $\mu=-\frac{1}{2}$ and final answer $\frac{13}{2}\mathbf{i}-\frac{3}{2}\mathbf{j}+\frac{1}{2}\mathbf{k}$, or equivalent [4]

(ii) Either equate scalar product of direction vector of AB and normal to q to zero or substitute for A and B in the equation of q and subtract expressions
Obtain $3+b-c=0$, or equivalent
Using the correct method for the moduli, divide the scalar product of the normals to p and q by the product of their moduli and equate to $\pm\frac{1}{2}$, or form horizontal

equivalent
Obtain correct equation in any form, e.g. $\dfrac{1+b}{\sqrt{(1+b^2+c^2)}\sqrt{(1+1)}}=\pm\dfrac{1}{2}$

Solve simultaneous equations for b or for c
Obtain $b=-4$ and $c=-1$
Use a relevant point and obtain final answer $x-4y-z=12$, or equivalent [7]
(The f.t. is on b and c.)

27. M/J 13/P31/Q6

(i) State or imply A is $(1, 4, -2)$
State or imply $\overrightarrow{QP}=12\mathbf{i}+6\mathbf{j}-6\mathbf{k}$ or equivalent
Use QP as normal and A as mid-point to find equation of plane
Obtain $12x+6y-6z=48$ or equivalent [4]

(ii) Either State equation of PB is $\mathbf{r}=7\mathbf{i}+7\mathbf{j}-5\mathbf{k}+\lambda\mathbf{i}$
Set up and solve a relevant equation for λ
Obtain $\lambda=-9$ and hence B is $(-2, 7, -5)$
Use correct method to find distance between A and B
Obtain 5.20

Or Obtain 12 for result of scalar product of QP and $\mathbf{i}$ or equivalent
Use correct method involving moduli, scalar product and cosine to find angle APB
Obtain $35.26°$ or equivalent
Use relevant trigonometry to find AB
Obtain 5.20 **[5]**

28. M/J 13/P33/Q10

(i) Equate scalar product of direction vector of l and p to zero
Solve for a and obtain $a=-6$ [2]

(ii) Express general point of l correctly in parametric form, e.g. $3\mathbf{i}+2\mathbf{j}+\mathbf{k}+\mu(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ or $(1-\mu)(3\mathbf{i}+2\mathbf{j}+\mathbf{k})+\mu(\mathbf{i}+\mathbf{j}-\mathbf{k})$
Equate at least two pairs of corresponding components of l and the second line and solve for λ or for μ

Obtain either $\lambda=\dfrac{2}{3}$ or $\mu=\dfrac{1}{3}$; or $\lambda=\dfrac{2}{a-1}$ or $\mu=\dfrac{1}{a-1}$; or reach $\lambda(a-4)=0$ or $(1+\mu)(a-4)=0$
Obtain $a=4$ having ensured (if necessary) that all three component equations are satisfied [4]

(iii) Using the correct process for the moduli, divide scalar product of direction vector of l and normal to p by the product of their moduli and equate to the sine of the given angle, or form an equivalent horizontal equation
Use $\frac{2}{\sqrt{5}}$ as sine of the angle

State equation in any form, e.g. $\dfrac{a+6}{\sqrt{(a^2+4+1)}\sqrt{(1+4+4)}}=\dfrac{2}{\sqrt{5}}$

Solve for a

Obtain answers for $a=0$ and $a=\dfrac{60}{31}$, or equivalent [5]

[Allow use of the cosine of the angle to score M1M1.]

29. O/N 12/P32/Q10,O/N 12/P31/Q10

(i) *EITHER* Use scalar product of relevant vectors, or subtract point equations to form two equations in a,b,c, e.g. $a-5b-3c=0$ and $a-b-3c=0$

State two correct equations in a,b,c

Solve simultaneous equations and find one ratio, e.g. $a:c$, or $b=0$

Obtain $a:b:c=3:0:1$, or equivalent

Substitute a relevant point in $3x+z=d$ and evaluate d

Obtain equation $3x+z=13$, or equivalent

OR 1 Attempt to calculate vector product of relevant vectors, e.g. $(\mathbf{i}-5\mathbf{j}-3\mathbf{k})\times(\mathbf{i}-\mathbf{j}-3\mathbf{k})$

Obtain 2 correct components of the product

Obtain correct product, e.g. $12\mathbf{i}+4\mathbf{k}$

Substitute a relevant point in $12x+4z=d$ and evaluate d

Obtain $3x+z=13$, or equivalent

OR 2 Attempt to form 2–parameter equation for the plane with relevant vectors

State a correct equation e.g. $\mathbf{r}=3\mathbf{i}-2\mathbf{j}+4\mathbf{k}+\lambda(\mathbf{i}-5\mathbf{j}-3\mathbf{k})+\mu(\mathbf{i}-\mathbf{j}-3\mathbf{k})$

State 3 equations in x, y, z, λ and μ

Eliminate λ and μ

Obtain equation $3x+z=13$, or equivalent [6]

(ii) *EITHER* Find $\overrightarrow{CP}$ for a point P on AB with a parameter t, e.g. $2\mathbf{i}+3\mathbf{j}+7\mathbf{k}+t(-\mathbf{i}+\mathbf{j}+3\mathbf{k})$

Either: Equate scalar product $\overrightarrow{CP}.\overrightarrow{AB}$ to zero and form an equation in t

Or 1: Equate derivative for CP^2 (or CP) to zero and form an equation in t

Or 2: Use Pythagoras in triangle CPA (or CPB) and form an equation in t

Solve and obtain correct value of t, e.g. $t=-2$

Carry out a complete method for finding the length of CP

Obtain answer $3\sqrt{2}$ (4.24), or equivalent

OR 1 State $\overrightarrow{AC}$ (or $\overrightarrow{BC}$) and $\overrightarrow{AB}$ in component form

Using a relevant scalar product find the cosine of CAB (or CBA)

Obtain cost $CAB=-\frac{22}{\sqrt{11}.\sqrt{62}}$, or $\cos CBA=\frac{33}{\sqrt{11}.\sqrt{117}}$, or equivalent

Use trig to find the length of the perpendicular

Obtain answer $3\sqrt{2}$ (4.24), or equivalent

OR 2 State $\overrightarrow{AC}$ (or $\overrightarrow{BC}$) and $\overrightarrow{AB}$ in component form

Using a relevant scalar product find the length of the projection AC (or BC) on AB

Obtain answer $2\sqrt{11}$ (or), $3\sqrt{11}$ or equivalent

Use Pythagoras to find the length of the perpendicular

Obtain answer $3\sqrt{2}$ (4.24), or equivalent

OR 3 State $\overrightarrow{AC}$ (or $\overrightarrow{BC}$) and $\overrightarrow{AB}$ in component form

Calculate their vector product, e.g. $(-2\mathbf{i}-3\mathbf{j}-7\mathbf{k})\times(-\mathbf{i}+\mathbf{j}+3\mathbf{k})$

Obtain correct product, e.g. $-2\mathbf{i}+13\mathbf{j}-5\mathbf{k}$

Divide modulus of the product by the modulus of $\overrightarrow{AB}$

Obtain answer $3\sqrt{2}$ (4.24), or equivalent

OR 4 State two of $\overrightarrow{AB}, \overrightarrow{BC}$) and $\overrightarrow{AC}$ in component form

Use cosine formula in triangle ABC to find $\cos A$ or $\cos B$

Obtain $\cos A=-\frac{44}{2\sqrt{11}.\sqrt{62}}$, or $\cos B=\frac{66}{2\sqrt{11}.\sqrt{117}}$

Use trig to find the length of the perpendicular [5]

Obtain answer $3\sqrt{2}$ (4.24), or equivalent

[The f.t is on $\overrightarrow{AB}$]

30. O/N 12/P33/Q8

(i) State or imply general point of either line has coordinates $(5+s, 1-s, -4+3s)$ or $(p+2t, 4+5t, -2-4t)$
Solve simultaneous equations and find s and t
Obtain $s = 2$ and $t = -1$ or equivalent in terms of p
Substitute in third equation to find $p = 9$
State point of intersection is $(7, -1, 2)$ [5]

(ii) Either Use scalar product to obtain a relevant equation in a, b, c
e.g. $a - b + 3c = 0$ or $2a + 5b - 4c = 0$
State two correct equations in a, b, c
Solve simultaneous equations to obtain at least one ratio
Obtain $a : b : c = -11 : 10 : 7$ or equivalent
Obtain equation $-11x + 10y + 7z = -73$ or equivalent with integer coefficients

Or 1 Calculate vector product of $\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\begin{pmatrix}2\\5\\-4\end{pmatrix}$

Obtain two correct components of the product

Obtain correct $\begin{pmatrix}-11\\10\\7\end{pmatrix}$ or equivalent

Substitute coordinates of a relevant point in $\mathbf{r.n} = d$ to find d

Obtain equation $-11x + 10y + 7z = -73$ or equivalent with integer coefficients

Or 2 Using relevant vectors, form correctly a two-parameter equation for the plane

Obtain $\mathbf{r} = \begin{pmatrix}5\\1\\-4\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\3\end{pmatrix} + \mu\begin{pmatrix}2\\5\\-4\end{pmatrix}$ or equivalent

State three equations in x, y, z, λ, μ
Eliminate λ and μ
Obtain $11x - 10y - 7z = 73$ or equivalent with integer coefficients [5]

31. M/J 12/P32/Q10

(i) *EITHER*: Substitute coordinates of a general point of l in given equation of plane m
Obtain equation in λ in any correct form
Verify that the equation is not satisfied for any value of λ

*OR*1: Substitute for $\mathbf{r}$ in the vector equation of plane m and expand scalar product
Obtain equation in λ in any correct form
Verify that the equation is not satisfied for any value of λ

*OR*2: Expand scalar product of a normal to m and a direction vector of l
Verify scalar product is zero
Verify that one point of l does not lie in the plane

*OR*3: Use correct method to find perpendicular distance of a general point of l from m
Obtain a correct unsimplified expression in terms of λ
Show that the perpendicular distance is 4/3, or equivalent, for all λ

*OR*4: Use correct method to find the perpendicular distance of a particular point of l from m
Obtain answer 4/3, or equivalent
Show that the perpendicular distance of a second point is also 4/3, or equivalent [3]

(ii) *EITHER*: Express general point of l in component form, e.g. $(1+2\lambda, 1+\lambda, -1+2\lambda)$
Substitute in given equation of n and solve for λ
Obtain position vector $5\mathbf{i}+3\mathbf{j}+3\mathbf{k}$ from $\lambda=2$

OR: State or imply plane n has vector equation $\mathbf{r}.(2\mathbf{i}-2\mathbf{j}+\mathbf{k})=7$, or equivalent
Substitute for $\mathbf{r}$, expand scalar product and solve for λ
Obtain position vector $5\mathbf{i}+3\mathbf{j}+3\mathbf{k}$ from $\lambda=2$ [3]

(iii) Form an equation in λ by equating perpendicular distances of a general point of l from m and n
Obtain a correct modular or non-modular equation in λ in any form
Solve for λ and obtain a point, e.g. $7\mathbf{i}+4\mathbf{j}+5\mathbf{k}$ from $\lambda=3$
Obtain a second point, e.g. $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ from $\lambda=1$
Use a correct method to find the distance between the two points
Obtain answer 6
[The f.t. is on the components of l.] [6]

32. M/J 12/P31/Q8

(i) Either Obtain $\pm\begin{pmatrix}2\\-1\\-15\end{pmatrix}$ for vector PA (where A is point on line) or equivalent

Use scalar product to find cosine of angle between PA and line

Obtain $\dfrac{42}{\sqrt{14\times230}}$ or equivalent

Use trigonometry to obtain $\sqrt{104}$ or 10.2 or equivalent

Or 1 Obtain $\pm\begin{pmatrix}2n+2\\n-1\\3n-15\end{pmatrix}$ for PN (where N is foot of perpendicular)

Equate scalar product of PN and line direction to zero
Or equate derivative of PN^2 to zero
Or use Pythagoras' theorem in triangle PNA to form equation in n
Solve equation and obtain $n=3$
Obtain $\sqrt{104}$ or 10.2 or equivalent

Or 2 Obtain $\pm\begin{pmatrix}2\\-1\\-15\end{pmatrix}$ for vector PA (where A is point on line)

Evaluate vector product of PA and line direction

Obtain $\pm\begin{pmatrix}12\\-36\\-4\end{pmatrix}$

Divide modulus of this by modulus of line direction and obtain $\sqrt{104}$ or 10.2 or equivalent

Or 3 Obtain $\pm\begin{pmatrix}2\\-1\\-15\end{pmatrix}$ for vector PA (where A is point on line)

Evaluate scalar product of PA and line direction to obtain distance AN
Obtain $3\sqrt{14}$ or equivalent
Use Pythagoras' theorem in triangle PNA and obtain $\sqrt{104}$ or 10.2 or equivalent

Or 4 Obtain $\pm\begin{pmatrix}2\\-1\\-15\end{pmatrix}$ for vector PA (where A is point on line)

Use a second point B on line and use cosine rule in triangle ABP to find angle A or angle B or use vector product to find area of triangle
Obtain correct answer (angle $A = 42.25\ldots$)
Use trigonometry to obtain $\sqrt{104}$ or 10.2 or equivalent [4]

(ii) Either Use scalar product to obtain a relevant equation in a, b, c, e.g. $2a+b+3c=0$ or $2a-b-15c=0$
State two correct equations in a, b and c
Solve simultaneous equations to obtain one ratio
Obtain $a:b:c=-3:9:-1$ or equivalent
Obtain equation $-3x+9y-z=28$ or equivalent

Or 1 Calculate vector product of two of $\begin{pmatrix}2\\1\\3\end{pmatrix}$, $\begin{pmatrix}2\\-1\\-15\end{pmatrix}$ and $\begin{pmatrix}8\\2\\-6\end{pmatrix}$ or equiv
Obtain two correct components of the product
Obtain correct $\begin{pmatrix}-3\\9\\-1\end{pmatrix}$ or equivalent
Substitute in $-3x+9y-z=d$ to find d or equivalent
Obtain equation $-3x+9y-z=28$ or equivalent

Or 2 Form a two-parameter equation of the plane
Obtain $\mathbf{r}=\begin{pmatrix}1\\3\\-4\end{pmatrix}+s\begin{pmatrix}2\\1\\3\end{pmatrix}+t\begin{pmatrix}2\\-1\\-15\end{pmatrix}$ or equivalent
State three equations in x, y, z, s, t
Eliminate s and t
Obtain equation $3x-9y+z=-28$ or equivalent [5]

33. M/J 12/P33/Q9

(i) Express general point of l or m in component form, i.e. $(3-\lambda, -2+2\lambda, 1+\lambda)$ or $(4+a\mu, 4+b\mu, 2-\mu)$
Equate components and eliminate either λ or μ from a pair of equations
Eliminate the other parameter and obtain an equation in a and b
Obtain the given answer [4]

(ii) Using the correct process equate the scalar product of the direction vectors to zero
Obtain $-a+2b-1=0$, or equivalent
Solve simultaneous equations for a or for b
Obtain $a=3$, $b=2$ [4]

(iii) Substitute found values in component equations and solve for λ or for μ
Obtain answer $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ from either $\lambda=2$ or from $\mu=-1$ [2]

34. O/N 11/P32/Q7, O/N 11/P31/Q7

(i) Use a correct method to express $\overrightarrow{OP}$ in terms of λ
Obtain the given answer [2]

(ii) *EITHER*: Use correct method to express scalar product of $\overrightarrow{OA}$ and $\overrightarrow{OP}$, or $\overrightarrow{OB}$ and $\overrightarrow{OP}$ in terms of λ
Using the correct method for the moduli, divide scalar products by products of

Obtain a correct simplified expression, e.g. $y = (31 - 7x) / 7$
Express the third variable in terms of the second
Obtain a correct simplified expression, e.g. $z = (3 - 3x) / 8$
Form a vector equation of the line
State a correct final answer, e.g. $\mathbf{r} = \frac{31}{8}\mathbf{j} + \frac{3}{8}\mathbf{k} + \lambda(-8\mathbf{i} + 7\mathbf{j} + 3\mathbf{k})$ [6]
[The f.t. is dependent on all M marks having been earned.]

37. M/J 11/P31/Q3

(i) Obtain $\pm\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}$ as normal to plane
Form equation of p as $3x - 4y + 6z = k$ or $-3x + 4y - 6z = k$ and use relevant point to find k
Obtain $3x - 4y + 6z = 80$ or $-3x + 4y - 6z = -80$ [3]

(ii) State the direction vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ or equivalent
Carry out correct process for finding scalar product of two relevant vectors
Use correct complete process with moduli and scalar product and evaluate $\sin^{-1}$ or $\cos^{-1}$ of result [4]
Obtain 30.8° or 0.538 radians

38. M/J 11/P33/Q10

(i) *EITHER*: Express general point of l or m in component form, e.g. $(2 + \lambda, -\lambda, 1 + 2\lambda)$ or $(\mu, 2 + 2\mu, 6 - 2\mu)$
Equate at least two pairs of components and solve for λ or for μ
Obtain correct answer for λ or μ (possible answers for λ are -2, $\frac{1}{4}$, 7 and for μ are 0, $2\frac{1}{4}$, $-4\frac{1}{2}$)
Verify that all three component equations are not satisfied

OR: State a relevant scalar triple product, e.g.
$(2\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}) \cdot ((\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}))$
Attempt to use the correct method of evaluation
Obtain at least two correct simplified terms of the three terms of the expansion of the triple product or of the corresponding determinant, e.g. $-4, -8, -15$
Obtain correct non-zero value, e.g. -27, and state that the lines do not intersect [4]

(ii) Carry out the correct process for evaluating scalar product of direction vectors for l and m
Using the correct process for the moduli, divide the scalar product by the product of the moduli and evaluate the inverse cosine of the result [3]
Obtain answer 47.1° or 0.822 radians

(iii) *EITHER*: Use scalar product to obtain $a - b + 2c = 0$
Obtain $a + 2b - 2c = 0$, or equivalent, from a scalar product, or by subtracting two point equations obtained from points on m, and solve for one ratio, e.g. $a : b$
Obtain $a : b : c = -2 : 4 : 3$, or equivalent
Substitute coordinates of a point on m and values for a, b and c in general equation and evaluate d
Obtain answer $-2x + 4y + 3z = 26$, or equivalent

OR1: Attempt to calculate vector product of direction vectors of l and m
Obtain two correct components
Obtain $-2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, or equivalent
Form a plane equation and use coordinates of a relevant point to evaluate d
Obtain answer $-2x + 4y + 3z = 26$, or equivalent

OR2: Form a two-parameter plane equation using relevant vectors
State a correct equation e.g. $\mathbf{r} = 2\mathbf{j} + 6\mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$
State three correct equations in x, y, z, s and t
Eliminate s and t
Obtain answer $-2x + 4y + 3z = 26$, or equivalent [5]

39. O/N 10/P33/Q6

(i) State general vector for point on line, e.g.
$-5\mathbf{i} + 3\mathbf{j} + 6\mathbf{k} + s(10\mathbf{i} + 5\mathbf{j} - 5\mathbf{k})$ or $5\mathbf{i} + 8\mathbf{j} + \mathbf{k} + t(10\mathbf{i} + 5\mathbf{j} - 5\mathbf{k})$ or equiv
Substitute their line into equation of plane and solve for parameter
Obtain correct value, $s = \frac{2}{5}$ or $t = -\frac{3}{5}$ or equivalent
Obtain $(-1, 5, 4)$ o.e. [4]

(ii) State or imply normal vector to p is $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$
Carry out process for evaluating scalar product of two relevant vectors
Using correct process for moduli, divide scalar product by the product of the moduli and evaluate arcsin(..) or arccos(..) of the result.
Obtain 5.1° or 0.089 rads [4]

40. O/N 09/P32/Q10

(i) Substitute coordinates $(1, 4, 2)$ in $2x - 3y + 6z = d$
Obtain plane equation $2x - 3y + 6z = 2$, or equivalent [2]

(ii) *EITHER*: Attempt to use plane perpendicular formula to find perpendicular from $(1, 4, 2)$ to p

Obtain a correct unsimplified expression, e.g. $\dfrac{|2 - 3(4) + 6(2) - 16|}{\sqrt{(2^2 + (-3)^2 + 6^2)}}$

Obtain answer 2

OR1: State or imply perpendicular from O to p is $\frac{16}{7}$, or from O to q is $\frac{2}{7}$, or equivalent
Find difference in perpendiculars
Obtain answer 2

OR2: Obtain correct parameter value, or position vector or coordinates of foot of perpendicular from $(1, 4, 2)$ to p ($\mu = \pm\frac{2}{7}$; $(\frac{11}{7}, \frac{22}{7}, \frac{26}{7})$)
Calculate the length of the perpendicular
Obtain answer 2

OR3: Carry out correct method for finding the projection onto a normal vector of a line segment joining a point on p, e.g. $(8, 0, 0)$ and a point on q, e.g. $(1, 4, 2)$

Obtain a correct unsimplified expression, e.g. $\dfrac{|2(8-1) - 3(-4) + 6(-2)|}{\sqrt{(2^2 + (-3)^2 + 6^2)}}$

Obtain answer 2 [3]

(iii) *EITHER*: Calling the direction vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, use scalar product to obtain a relevant equation in a, b and c
Obtain two correct equations, e.g. $2a - 3b + 6c = 0$, $a - 2b + 2c = 0$
Solve for one ratio, e.g. $a : b$
Obtain $a : b : c = 6 : 2 : -1$, or equivalent
State answer $\mathbf{r} = \lambda(6\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ or equivalent

OR: Attempt to calculate vector product of two normals, e.g.
$(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$
Obtain two correct components
Obtain $-6\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, or equivalent
State answer $\mathbf{r} = \lambda(-6\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, or equivalent [5]

41. O/N 09/P31/Q6

(i) *EITHER*: State that the position vector of M is $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, or equivalent
Carry out a correct method for finding the position vector of N
Obtain answer $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, or equivalent
Obtain vector equation of MN in any correct form,
e.g. $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

OR: State that the position vector of M is $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, or equivalent
Carry out a correct method for finding a direction vector for MN
Obtain answer, e.g. $\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, or equivalent
Obtain vector equation of MN in any correct form, [4]
e.g. $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$
[SR: The use of $AN = AC/3$ can earn M1A0, but $AN = AC/2$ gets M0A0.]

(ii) State equation of BC in any correct form, e.g. $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \mu(\mathbf{i} - 5\mathbf{j} + 5\mathbf{k})$
Solve for λ or for μ [4]
Obtain correct value of λ, or μ, e.g. $\lambda = 3$, or $\mu = 2$
Obtain position vector $5\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}$

42. O/N 08/P03/Q7

(i) State or imply a correct normal vector to either plane, e.g. $2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$, or $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
Carry out correct process for evaluating the scalar product of the two normals
Using the correct process for the moduli, divide the scalar product by the product of the moduli [4]
and evaluate the inverse cosine of the result
Obtain answer 57.7° (or 1.01 radians)

(ii) *EITHER*: Carry out a complete method for finding a point on the line
Obtain such a point, e.g. $(2, 0, -1)$

EITHER: State two correct equations for a direction vector of the line, e.g. $2a - b - 3c = 0$
and $a + 2b + 2c = 0$
Solve for one ratio, e.g. $a : b$
Obtain $a : b : c = 4 : -7 : 5$, or equivalent
State a correct answer, e.g. $\mathbf{r} = 2\mathbf{i} - \mathbf{k} + \lambda(4\mathbf{i} - 7\mathbf{j} + 5\mathbf{k})$

OR: Obtain a second point on the line, e.g. $(0, \frac{7}{2}, -\frac{7}{2})$
Subtract position vectors to obtain a direction vector
Obtain $4\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$, or equivalent
State a correct answer, e.g. $\mathbf{r} = 2\mathbf{i} - \mathbf{k} + \lambda(4\mathbf{i} - 7\mathbf{j} + 5\mathbf{k})$

OR: Attempt to calculate the vector product of two normals
Obtain two correct components
Obtain $4\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$, or equivalent
State a correct answer, e.g. $\mathbf{r} = 2\mathbf{i} - \mathbf{k} + \lambda(4\mathbf{i} - 7\mathbf{j} + 5\mathbf{k})$

OR1: Express one variable in terms of a second
Obtain a correct simplified expression, e.g. $x = \dfrac{14 - 4y}{7}$
Express the first variable in terms of a third
Obtain a correct simplified expression, e.g. $x = \dfrac{14 + 4z}{5}$
Form a vector equation for the line
State a correct answer, e.g. $\mathbf{r} = \frac{7}{2}\mathbf{j} - \frac{7}{2}\mathbf{k} + \lambda(\mathbf{i} - \frac{7}{4}\mathbf{j} + \frac{5}{4}\mathbf{k})$, or equivalent

OR2: Express one variable in terms of a second
Obtain a correct simplified expression, e.g. $y = \frac{14-7x}{4}$
Express the third variable in terms of the second
Obtain a correct simplified expression, e.g. $z = \frac{5x-14}{4}$
Form a vector equation for the line
State a correct answer, e.g. $\mathbf{r} = \frac{7}{2}\mathbf{j} - \frac{7}{2}\mathbf{k} + \lambda(\mathbf{i} - \frac{7}{4}\mathbf{j} + \frac{5}{4}\mathbf{k})$, or equivalent **[6]**
[The f.t. is dependent on all M marks having been obtained.]

43. M/J 08/P03/Q10

(i) State a vector equation for the line through A and B, e.g. $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + s(\mathbf{i} - \mathbf{j})$
Equate at least two pairs of components of general points on AB and l, and solve for s or for t
Obtain correct answer for s or t, e.g. $s = -6, 2, -2$ when $t = 3, -1, -1$ respectively
Verify that all three component equations are not satisfied [4]

(ii) State or imply a direction vector for AP has components $(-2t, 3+t, -1-t)$, or equivalent
State or imply cos 60° equals $\frac{\overrightarrow{AP}.\overrightarrow{AB}}{|\overrightarrow{AP}||\overrightarrow{AB}|}$
Carry out correct processes for expanding the scalar product and expressing the product of the moduli in terms of t, in order to obtain an equation in t in any form
Obtain the given equation $3t^2 + 7t + 2 = 0$ correctly
Solve the quadratic and use a root to find a position vector for P
Obtain position vector $5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ from $t = -2$, having rejected the root $t = -\frac{1}{3}$ for a valid reason [6]

44. O/N 07/P03/Q10

(i) Substitute for $\mathbf{r}$ and expand the given scalar product, or correct equivalent, to obtain an equation in s
Solve a linear equation formed from a scalar product for s
Obtain $s = 2$ and position vector $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ for A **[3]**

(ii) State or imply a normal vector of p is $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$, or equivalent
Use the correct process for evaluating a relevant scalar product, e.g. $(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}).(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$
Using the correct process for calculating the moduli, divide the scalar product by the product of the moduli and evaluate the inverse sine or cosine of the result
Obtain final answer 72.2° or 1.26 radians [4]

(iii) *EITHER*: Taking the direction vector of the line to be $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, state equation $2a - 3b + 6c = 0$
State equation $a - 2b + 2c = 0$
Solve to find one ratio, e.g. $a : b$
Obtain ratio $a : b : c = 6 : 2 : -1$, or equivalent
State answer $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(6\mathbf{i} + 2\mathbf{j} - \mathbf{k})$, or equivalent

OR1: Attempt to calculate the vector product of a direction vector for the line l and a normal vector of the plane p, e.g. $(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$
Obtain two correct components of the product
Obtain answer $-6\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, or equivalent
State answer $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(-6\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, or equivalent

OR2: Obtain the equation of the plane containing A and perpendicular to the line l
State answer $x - 2y + 2z = 1$, or equivalent
Find position vector of a second point B on the line of intersection of this plane with the plane p, e.g. $9\mathbf{i} + 4\mathbf{j}$
Obtain a direction vector for this line of intersection, e.g. $6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
State answer $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(6\mathbf{i} + 2\mathbf{j} - \mathbf{k})$, or equivalent [5]
[The f.t. is on A.]

45. M/J 07/P03/Q7

(i) State or imply $du = \frac{1}{2\sqrt{x}}dx$, or $2u\,du - dx$, or $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, or equivalent

Substitute for x and dx throughout the integral

Obtain the given form of indefinite integral correctly with no errors seen 3

(ii) Attempting to express the integrand as $\frac{A}{u} + \frac{B}{4-u}$, use a correct method to find either A or B

Obtain $A = \frac{1}{2}$ and $B = \frac{1}{2}$

Integrate and obtain $\frac{1}{2}\ln u - \frac{1}{2}\ln(4-u)$, or equivalent

Use limits u = 1 and u = 2 correctly, or equivalent, in an integral of the form $c\ln u + d\ln(4-u)$

Obtain given answer correctly following full and exact working.

46. O/N 06/P03/Q7

(i) EITHER: State or imply general point of l has coordinates (x, 1 – 2x, 1+s), or equivalent

Substitute in LHS of plane equation

Verify that the equation is satisfied

OR: State or imply the plane has equation $r.(i + 2j + 3k) = 5$, or equivalent

Substitute for r in LHS and expand the scalar product

Verify that the equation is satisfied

OR: Verify that a point of l lies on the plane

Find a second point on l and substitute its coordinates in the equation of P

Verify second point, e.g. (1, – 1, 2) lies on the plane

OR: Verify that a point of l lies on the plane

Form scalar product of a direction vector of l with a vector normal to p [3]

Verify scalar product is zero and l is parallel to p

(ii) EITHER: Use scalar product of relevant vectors to form an equation in a, b, c, e.g, $a - 2b + c = 0$ or $a + 2b + 3c = 0$

State two correct equations in a, b, c

Solve simultaneous equations and find one ratio, e.g a : b

Obtain a : b : c = 4 : 1 : –2, or equivalent

Substitute correctly in $4x + y - 2z = d$ to find d

Obtain equation $4x + y - 2z = 1$, or equivalent.

OR: Attempt to calculate vector product of relevant vectors, e.g. $(i - 2j + k) \times (i + 2j + 3k)$

Obtain 2 correct components of the product

Obtain correct product, e.g. $-8i - 2j + 4k$

Substitute correctly in $4x + y - 2z = d$ to find d

Obtain equation $4x + y - 2z = 1$ or equivalent

[SR: If the outcome of the vector product is the negative of the correct answer allow the final marks to be available, i.e M2A0A0M1A1 is possible.]

OR: Attempt to form 2–parameter equation for the plane with relevant vectors

State a correct equation, e.g. $r = 2i + j + 4k + \lambda(i - 2j + k) + \mu(i + 2j + 3k)$

State 3 equations in x, y, z, λ, μ

Eliminate λ and μ [6]

Obtain equation $4x + y - 2z = 1$, or equivalent.

47. M/J 06/P03/Q10

(i) State $r = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}$, or equivalent [1]

(ii) Express $\overrightarrow{BN}$ in terms of λ, e.g. $\begin{pmatrix}-1+3\lambda\\3-\lambda\\5-4\lambda\end{pmatrix}-\begin{pmatrix}3\\-1\\-4\end{pmatrix}$, or equivalent

Equate its scalar product with $\begin{pmatrix}3\\-1\\-4\end{pmatrix}$ to zero and solve for λ

Obtain $\lambda=2$

Obtain $\overrightarrow{ON}=\begin{pmatrix}5\\1\\-3\end{pmatrix}$, or equivalent

Carry out method for calculating BN, i.e. $|2i+2j+k|$

Obtain the given answer BN = 3 correctly [6]

(iii) EITHER: Use scalar product to obtain a relevant equation in a, b and c, e.g 3a – b –4c = 0 or 2a + 2b + c = 0
State two correct equations in a, b, c
Solve simultaneous equations to obtain one ratio, e.g. a : b
Obtain a : b : c = 7 : –11 : 8, or equivalent
Obtain equation 7x – 11y + 8z = 0, or equivalent

OR: Substitute for A, B and N in equation of plane and state 3 equations in a, b, c, d
Eliminate one unknown, e.g. d, entirely and obtain 2 equations in 3 unknowns
Solve to obtain one ratio e.g. a : b
Obtain a : b : c = 7 : –11 : 8, or equivalent
Obtain equation 7x – 11y + 8z = 0, or equivalent

OR: Calculate vector product of two vectors parallel to the plane, e.g.(3i – j – 4k)×(2i + 2j + k)
Obtain 2 correct components of the product
Obtain correct product e.g. 7i – 11j + 8k, or equivalent
Substitute equation 7x – 11y + 8z = d and find d, or equivalent
Obtain equation 7x – 11y + 8z = 0, or equivalent

OR; Form correctly a 2-parameter equation for the plane
Obtain equation in any correct form e.g. r = –i + 3j + 5k + λ (3i – j – 4k) + μ (2i + 2j + k)
State 3 equations in x, y, z, λ, and μ
Eliminate λ and μ
Obtain equation 7x – 11y + 8z = 0 or equivalent. [5]

48. O/N 05/P03/Q10

(i) State or imply a direction vector of *AB* is –**i** + 2**j** + **k**, or equivalent
State equation of *AB* is **r** = 2**i** + 2**j** + **k** + λ(–**i** + 2**j** + **k**), or equivalent
Substitute in equation of *p* and solve for λ
Obtain 4**i** – 2**j** – **k** as position vector of *C* [4]

(ii) State or imply a normal vector of *p* is **i** – 2**j** + 2**k**, or equivalent
Carry out correct process for evaluating the scalar product of two relevant vectors, e.g. (–**i** + 2**j** + **k**).(**i** – 2**j** + 2**k**)
Using the correct process for calculating the moduli, divide the scalar product by the product of the moduli and evaluate the inverse cosine or inverse sine of the result
Obtain answer 24.1° [4]

(iii) *EITHER*: Obtain *AC* ($=\sqrt{24}$) in any correct form
Use trig to obtain length of perpendicular from *A* to *p*
Obtain given answer correctly

OR: State or imply $\overrightarrow{AC}$ is $2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$, or equivalent
Use scalar product of $\overrightarrow{AC}$ and a unit normal of p to calculate the perpendicular
Obtain given answer correctly

OR: Use plane perpendicular formula to find perpendicular from A to p
Obtain a correct unsimplified numerical expression, e.g. $\dfrac{|2-2(2)+2(1)-6|}{\sqrt{(1^2+(-2)^2+2^2)}}$

Obtain given answer correctly [3]

49. M/J 05/P03/Q10

(i) State or imply a direction vector for AB is $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, or equivalent

EITHER: State equation of AB is $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$, or equivalent
Equate at least two pairs of components of AB and l and solve for s or for t
Obtain correct answer for s or for t, e.g. $s = 0$ or $t = -2$; $s = -\frac{5}{3}$ or $t = -\frac{1}{3}$ or $s = 5$ or $t = 3$
Verify that all three pairs of equations are not satisfied and that the lines fail to intersect

OR: State a Cartesian equation for AB, e.g. $\dfrac{x-2}{-1} = \dfrac{y-2}{2} = \dfrac{z-1}{2}$, and for l,
e.g. $\dfrac{x-4}{1} = \dfrac{y+2}{2} = \dfrac{z-2}{1}$
Solve a pair of equations, e.g. in x and y, for one unknown
Obtain one unknown, e.g. $x = 4$ or $y = -2$
Obtain corresponding remaining values, e.g. of z, and show lines do not intersect

OR: Form a relevant triple scalar product,
e.g. $(2\mathbf{i} - 4\mathbf{j} + \mathbf{k}).((-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})\times(\mathbf{i} + 2\mathbf{j} + \mathbf{k}))$
Attempt to use correct method of evaluation
Obtain at least two correct simplified terms of the three terms of the complete expansion of the triple product or of the corresponding determinant
Obtain correct non-zero value, e.g. -20, and state that the lines do not intersect 5

(ii)

EITHER: Obtain a vector parallel to the plane and not parallel to l, e.g. $2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$
Use scalar product to obtain an equation in a, b and c, e.g. $a + 2b + c = 0$
Form a second relevant equation, e.g. $2a - 4b + c = 0$ and solve for one ratio, e.g. $a : b$
Obtain final answer $a : b : c = 6 : 1 : -8$
Use coordinates of a relevant point and values of a, b and c in general equation and find d
Obtain answer $6x + y - 8z = 6$, or equivalent

OR: Obtain a vector parallel to the plane and not parallel to l, e.g. $2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$
Obtain a second relevant vector parallel to the plane and attempt to calculate their vector product, e.g. $(\mathbf{i} + 2\mathbf{j} + \mathbf{k})\times(2\mathbf{i} - 4\mathbf{j} + \mathbf{k})$
Obtain two correct components of the product
Obtain correct answer, e.g. $6\mathbf{i} + \mathbf{j} - 8\mathbf{k}$
Substitute coordinates of a relevant point in $6x + y - 8z = d$, or equivalent, to find d
Obtain answer $6x + y - 8z = 6$, or equivalent

OR: Obtain a vector parallel to the plane and not parallel to l, e.g. $2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$
Obtain a second relevant vector parallel to the plane and correctly form a 2-parameter equation for the plane,
e.g. $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - 4\mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
State 3 correct equations in x, y, z, λ and μ
Eliminate λ and μ
Obtain equation in any correct form
Obtain answer $6x + y - 8z = 6$, or equivalent

OR: Using the coordinates of A and two points on l, state three simultaneous equations in a, b, c and d, e.g. $2a + 2b + c = d$, $4a - 2b + 2c = d$ and $5a + 3c = d$
Solve and find one ratio, e.g. $a:b$
State one correct ratio
Obtain a ratio of three unknowns, e.g. $a:b:c = 6:1:-8$, or equivalent
Either use coordinates of a relevant point and found ratio to find fourth unknown, e.g. d, **or** find the ratio of all four unknowns
Obtain answer $6x + y - 8z = 6$, or equivalent **6**

50. O/N 04/P03/Q9

(i) *EITHER*: Express general point of l or m in component form
e.g. $(2 + s, -1 + s, 4 - s)$ or $(-2 - 2t, 2 + t, 1 + t)$
Equate at least two pairs of components and solve for s or for t
Obtain correct answer for s or t (possible answers are $\frac{2}{3}$, 10, or 3 for s and $-\frac{7}{3}$, -7, or 0 for t)
Verify that all three component equations are not satisfied

OR: State a Cartesian equation for l or for m, e.g. $\frac{x-2}{1} = \frac{y-(-1)}{1} = \frac{z-4}{-1}$ for l
Solve a pair of equations for a pair of values, e.g. x and y
Obtain a pair of correct answers, e.g. $x = \frac{8}{3}$ and $y = -\frac{1}{3}$
Find corresponding remaining values, e.g. of z, and show lines do not intersect

OR: Form a relevant triple scalar product, e.g.
$(4\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}).((\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (-2\mathbf{i} + \mathbf{j} + \mathbf{k}))$
Attempt to use correct method of evaluation
Obtain at least two correct simplified terms of the three terms of the complete expansion of the triple product or of the corresponding determinant
Obtain correct non-zero value, e.g. 14, and state that the lines cannot intersect **4**

(ii) *EITHER*: Express $\overrightarrow{PQ}$ or $(\overrightarrow{QP})$ in terms of s in any correct form e.g.
$-s\mathbf{i} + (1 - s)\mathbf{j} + (-5 + s)\mathbf{k}$
Equate its scalar product with a direction vector for l to zero, obtaining a linear equation in s
Solve for s
Obtain $s = 2$ and $\overrightarrow{OP}$ is $4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

OR: Take a point A on l, e.g. $(2, -1, 4)$, and use scalar product to calculate AP, the length of the projection of AQ onto l
Obtain answer $AP = 2\sqrt{3}$, or equivalent
Carry out method for finding $\overrightarrow{OP}$
Obtain answer $4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ **4**

(iii). Show that Q is the point on m with parameter $t = -2$, or that $(2, 0, -1)$ satisfies the Cartesian equation of m
Show that PQ is perpendicular to m e.g. by verifying fully that
$(-2\mathbf{i} - \mathbf{j} - 3\mathbf{k}).(-2\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0$ 2

51. M/J 04/P03/Q11

(i) *EITHER*: Obtain a vector in the plane e.g. $\overrightarrow{PQ} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$
Use scalar product to obtain a relevant equation in a, b, c e.g. $-3a + 4b + c = 0$ or $6a - 2b + c = 0$ or $3a + 2b + 2c = 0$
State two correct equations in a, b, c
Solve simultaneous equations to obtain one ratio e.g. $a : b$
Obtain $a : b : c = 2 : 3 : -6$ or equivalent
Obtain equation $2x + 3y - 6z$
[The second M1 is also given if say c is given an arbitrary value and a or b is found. The following A1 is then given for finding the correct values of a and b.]

OR: Substitute for P, Q, R in equation of plane and state 3 equations in a, b, c, d
Eliminate one unknown, e.g. d, entirely
Obtain 2 equations in 3 unknowns
Solve to obtain one ratio e.g. $a : b$
Obtain $a : b : c = 2 : 3 : -6$ or equivalent
Obtain equation $2x + 3y - 6z = 8$ or equivalent
[The first M1 is also given if say d is given an arbitrary value and two equations in two unknowns, e.g. a and b, are obtained. The following A1 is for two correct equations. Solving to obtain one unknown earns the second M1 and the following A1 is for finding the correct values of a and b.]

OR: Obtain a vector in the plane e.g. $\overrightarrow{QR} = 6\mathbf{i} - 2\mathbf{j} + \mathbf{k}$
Find a second vector in the plane and form correctly a 2-parameter equation for the plane
Obtain equation in any correct form e.g. $\mathbf{r} = \lambda(-3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + \mu(6\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + \mathbf{i} - \mathbf{k}$
State 3 equations in x, y, z, λ, and μ
Eliminate λ and μ
Obtain equation $2x + 3y - 6z = 8$ or equivalent

OR: Obtain a vector in the plane e.g. $\overrightarrow{PR} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
Obtain a second vector in the plane and calculate the vector product of the two vectors, e.g. $(-3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) \times (3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$
Obtain 2 correct components of the product
Obtain correct product e.g. $6\mathbf{i} + 9\mathbf{j} - 18\mathbf{k}$ or equivalent
Substitute in $2x + 3y - 6z = d$ and find d or equivalent
Obtain equation $2x + 3y - 6z = 8$ or equivalent 6

(ii) *EITHER*: State equation of SN is $\mathbf{r} = 3\mathbf{i} + 5\mathbf{j} - 6\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$ or equivalent
Express x, y, z in terms of λ e.g. $(3 + 2\lambda, 5 + 3\lambda, -6 - 6\lambda)$
Substitute in the equation of the plane and solve for λ
Obtain $\overrightarrow{ON} = \mathbf{i} + 2\mathbf{j}$, or equivalent
Carry out method for finding SN
Show that $SN = 7$ correctly

OR: Letting $\overrightarrow{ON} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, obtain two equations in x, y, z by equating scalar product of $\overrightarrow{NS}$ with two of $\overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{RP}$ to zero
Using the plane equation as third equation, solve for x, y and z
Obtain $\overrightarrow{ON} = \mathbf{i} + 2\mathbf{j}$, or equivalent
Carry out method for finding SN
Show that $SN = 7$ correctly

OR: Use Cartesian formula or scalar product of $\overrightarrow{PS}$ with a normal vector to find SN
Obtain $SN = 7$
State a unit normal $\hat{\mathbf{n}}$ to the plane
Use $\overrightarrow{ON} = \overrightarrow{OS} \pm 7\hat{\mathbf{n}}$

Obtain an unsimplified expression e.g. $3i + 5j - 6k \pm 7(\frac{2}{7}i + \frac{3}{7}j - \frac{6}{7}k)$
Obtain $\overrightarrow{ON} = i + 2j$, or equivalent, only 6

52. O/N 03/P03/Q10

(i) Express general point of l or m in component form e.g. $(1 + 2s, s, -2 + 3s)$ or $(6 + t, -5 - 2t, 4 + t)$
Equate at least two corresponding pairs of components and attempt to solve for s or t
Obtain $s = 1$ or $t = -3$
Verify that all three component equations are satisfied
Obtain position vector $3\mathbf{i} + \mathbf{j} + \mathbf{k}$ of intersection point, or equivalent [5]

(ii) *EITHER:* Use scalar product to obtain $2a + b + 3c = 0$ and $a - 2b + c = 0$
Solve and find one ratio e.g. $a : b$
State one correct ratio
Obtain answer $a : b : c = 7 : 1 : -5$, or equivalent
Substitute coordinates of a relevant point and values of a, b and c in general equation of plane and calculate d
Obtain answer $7x + y - 5z = 17$, or equivalent

OR: Using two points on l and one on m (or *vice versa*) state three simultaneous equations in a, b, c and d e.g. $3a + b + c = d$, $a - 2c = d$ and $6a - 5b + 4c = d$
Solve and find one ratio e.g. $a : b$
State one correct ratio
Obtain a ratio of three unknowns e.g. $a : b : c = 7 : 1 : -5$, or equivalent
Use coordinates of a relevant point and found ratio to find fourth unknown e.g. d
Obtain answer $7x + y - 5z = 17$, or equivalent

OR: Form a correct 2-parameter equation for the plane,
e.g. $\mathbf{r} = \mathbf{i} - 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$
State 3 equations in x, y, z, λ and μ
State 3 correct equations
Eliminate λ and μ
Obtain equation in any correct unsimplified form
Obtain $7x + y - 5z = 17$, or equivalent

OR: Attempt to calculate vector product of vectors parallel to l and m
Obtain two correct components of the product
Obtain correct product, e.g. $7\mathbf{i} + \mathbf{j} - 5\mathbf{z}$
State that the plane has equation of the form $7x + y - 5z = d$
Substitute coordinates of a relevant point and calculate d
Obtain answer $7x + y - 5z = 17$, or equivalent

[The follow through is on $3\mathbf{i} + \mathbf{j} + \mathbf{k}$ only.] [6]

53. M/J 03/P03/Q9

(i) State or imply a correct normal vector to either plane,
e.g. $i + 2j - 2k$ or $2i - 3j + 6k$
Carry out correct process for evaluating the scalar product of both the normal vectors
Using the correct process for the moduli, divide the scalar product of the two normals by the product of their moduli and evaluate the inverse cosine of the result
Obtain answer 40.4° (or 40.3°) or 0.705 (or 0.704) radians
[Allow the obtuse answer 139.6° or 2.44 radians] [4]

(ii) *EITHER* Carry out a complete strategy for finding a point on l
Obtain such a point e.g. (0, 3, 2)

EITHER Set up two equations for a direction vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ of l, e.g. $a + 2b - 2c = 0$ and $2a - 3b + 6c = 0$
Solve for one ratio, e.g. $a:b$
Obtain $a:b:c = 6:-10:-7$, or equivalent
State a correct answer, e.g. $\mathbf{r} = 3\mathbf{j} + 2\mathbf{k} + \lambda(6\mathbf{i} - 10\mathbf{j} - 7\mathbf{k})$

OR Obtain a second point on l, e.g. (6, -7, -5)
Subtract position vectors to obtain a direction vector for l
Obtain $6\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}$, or equivalent
State a correct answer, e.g. $\mathbf{r} = 3\mathbf{j} + 2\mathbf{k} + \lambda(6\mathbf{i} - 10\mathbf{j} - 7\mathbf{k})$

OR Attempt to find the vector product of the two normal vectors
Obtain two correct components
Obtain $6\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}$, or equivalent
State a correct answer, e.g. $\mathbf{r} = 3\mathbf{j} + 2\mathbf{k} + \lambda(6\mathbf{i} - 10\mathbf{j} - 7\mathbf{k})$

OR Express one variable in terms of a second
Obtain a correct simplified expression, e.g. $x = (9 - 3y)/5$
Express the same variable in terms of the third and form a three term equation
Incorporate a correct simplified expression, e.g. $x = (12 - 6z)/7$ in this equation
Form a vector equation for the line
State a correct answer, e.g. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -5/3 \\ -7/6 \end{pmatrix}\lambda$, or equivalent

OR Express one variable in terms of a second, e.g. $y = (9 - 5x)/3$
Obtain a correct simplified expression, e.g. $y = (9 - 5x)/3$
Express the third variable in terms of the second
Obtain a correct simplified expression, e.g. $z = (12 - 7x)/6$
Form a vector equation for the line
State a correct answer, e.g. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -5/3 \\ -7/6 \end{pmatrix}$, or equivalent **[6]**

54. O/N 02/P03/Q10

(i) Find a direction vector for AB or CD e.g. $\overrightarrow{AB} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ or $\overrightarrow{CD} = -2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$

EITHER: Carry out the correct process for evaluating the scalar product of two relevant vectors in component form
Evaluate $\cos^{-1}\left(\frac{\overrightarrow{AB}.\overrightarrow{CD}}{|\overrightarrow{AB}|.|\overrightarrow{CD}|}\right)$ using the correct method for the moduli
Obtain final answer 45.6 °, or 0.796 radians, correctly

OR: Calculate the sides of a relevant triangle using the correct method
Use the cosine rule to calculate a relevant angle
Obtain final answer 45.6 °, or 0.796 radians, correctly
[SR: if a vector is incorrectly stated with all signs reversed and 45.6 ° is obtained, award B0M1M1A1.]
[SR: if 45.6 ° is followed by 44.4 ° as final answer, award A0.] **[4]**

(ii) EITHER: State both line equations e.g. $4i + k + \lambda(i - 2j - 3k)$ and $i + j + u(2i + j + 4k)$

Equate components and solve for λ or for μ

Obtain value $\lambda = -1$ or $\mu = 1$

Verify that all equations are satisfied, so that the lines do intersect, or equivalent

[SR: if both lines have the same parameter, award B1M1 if the equations are inconsistent and B1M1A1 if the equations are consistent and shown to be so.]

OR: State both line equations in Cartesian form

Solve simultaneous equations for a pair of unknowns e.g. x and y

Obtain a correct pair e.g. $x = 3, y = 2$

Obtain the third unknown e.g. $z = 4$ and verify the lines intersect

OR; Find one of $\overrightarrow{CA}, \overrightarrow{CB}, \overrightarrow{DA}, \overrightarrow{DB}, \ldots\ldots,$ e.g. $\overrightarrow{CA} = 3i - j + k$

Carry out correct process for evaluating a relevant scalar triple product e.g. $\overrightarrow{CA}, (\overrightarrow{AB} \times \overrightarrow{CD})$

Show the value is zero

State that (a) this result implies the lines are coplanar, (b) the lines are not parallel, and thus the lines intersect (condone omission of one of (a) and (b))

OR: Carry out correct method for finding a normal to the plane through three of the points

Obtain a correct normal vector

Obtain a correct equation e.g. $x + 2y - z = 3$ for the plane of A, B, C

Verify that the fourth point lies in the plane and conclude that the lines intersect

OR: State a relevant plane equation e.g. $r = 4i + k + \lambda(i - 2j - 3k) + \mu(-3i + j - k)$ for the Plane of A, B, C

Set up equations in λ and μ, using components of the fourth point, and solve for λ or μ

Obtain value $\lambda = 1$ or $\mu = 2$

Verify that all equations are satisfied and conclude that the lines intersect **[4]**

(iii) EITHER : Find $\overrightarrow{PQ}$ for a general point Q on AB e.g. $3i - 5j - 5k + \lambda(i - 2j - 3k)$

Calculate $\overrightarrow{PQ} . \overrightarrow{AB}$ correctly and equate to zero

Solve for λ obtaining $\lambda = -2$

Show correctly that $PQ = \sqrt{3}$, the given answer

OR: State $\overrightarrow{AP}$ (or $\overrightarrow{BP}$)and $\overrightarrow{AB}$ in component form

Carry out correct method for finding their vector product

Obtain correct answer e.g. $\overrightarrow{AP} \times \overrightarrow{AB} = -5i - 4j + k$

Divide modulus by $|\overrightarrow{AB}|$ and obtain the given answer $\sqrt{3}$

OR: State $\overrightarrow{AP}$ (or $\overrightarrow{BP}$) and $\overrightarrow{AB}$ in component form

Carry out correct method for finding the projection of AP(or BP) on AB i.e. $\dfrac{\overrightarrow{AP}.\overrightarrow{AB}}{|\overrightarrow{AB}|}$

Obtain correct answer e.g. $AN = \dfrac{28}{\sqrt{14}}$ or $BN = \dfrac{42}{\sqrt{14}}$

Show correctly that $PN = \sqrt{3}$, the given answer

OR: State two of $\overrightarrow{AP}, \overrightarrow{BP}, \overrightarrow{AB}$ in component form

Use the cosine rule in triangle ABP, or scalar product, to find the cosine of A, B, or P

Obtain correct answer e.g. $\cos A = \dfrac{-28}{\sqrt{14}.\sqrt{59}}$

Deduce the exact length of the perpendicular from P to AB is $\sqrt{3}$, the given answer

[4]

55. M/J 02/P03/Q8

(i) State or imply a simplified direction vector of l is $3i - j + 2k$, or equivalent

State equation of l is $r = i + k + \lambda(3i - j + 2k)$, or $\dfrac{x-1}{3} = \dfrac{y}{-1} = \dfrac{z-1}{2}$, or equivalent

Substitute in equation of P and solve for λ, or one of x, y, or z

Obtain point of intersection $-2i + j - k$ [4]

[Any notation is acceptable.]

(ii) State or imply a normal vector of p is $i + 3j - 2k$

EITHER : Use scalar product to obtain $a + 3b - 2c = 0$

Use points on l to obtain two equations in a, b, c e.g. $a + c = 1$, $4a - b + 3c = 1$

Solve simultaneous equations, obtaining one unknown

Obtain one correct unknowns e.g. $a = \dfrac{-2}{3}$

Obtain the other unknowns e.g. $b = \dfrac{4}{3}$, $c = \dfrac{5}{3}$

OR: Use scalar product to obtain $a + 3b - 2c = 0$

Use scalar product to obtain $3a - b + 2c = 0$

Solve simultaneous equations to obtain one ratio e.g. a : b

Obtain $a : b : c = 2 : -4 : -5$, or equivalent

Obtain $a = \dfrac{-2}{3}$, $b = \dfrac{4}{3}$, $c = \dfrac{5}{3}$

[NB : candidates may transfer from the EITHER to OR scheme by subtracting the two "point" equations, or transfer from OR to EITHER by finding one of the "point" equation.]

OR: Calculate the vector product $(3i - j + 2k) \times (i + 3j - 2k)$

Obtain answer $-4i + 8j + 10k$, or equivalent

Substitute in $-4x + 8y + 10z = d$ to find d, or equivalent

Obtain $d = 6$, or equivalent

Obtain $a = \dfrac{-2}{3}$, $b = \dfrac{4}{3}$, $c = \dfrac{5}{3}$

OR: State or imply a correct equation of the plane e.g. $r = \lambda(3i - j + 2k) + \mu(i + 3j - 2k) + i + k$

State 3 equations in x, y, z, λ, and μ, e.g. $x = 3\lambda + \mu + 1$, $y = -\lambda + 3\mu$, $z = 2\lambda - 2\mu + 1$

Eliminate λ and μ

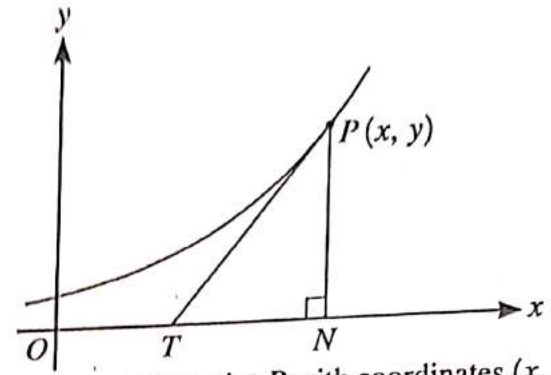
Obtain equation $-4x + 8y + 10z = 6$, or equivalent

Obtain $a = -\dfrac{2}{3}$, , $b = \dfrac{4}{3}$, $c = \dfrac{5}{3}$ [6]

[SR : condone the use of $xi + yj + zk$ for $ai + bj + ck$ in the EITHER scheme and the first OR scheme]

Unit-8: Differential Equations

1. M/J 18/P32/Q3



In the diagram, the tangent to a curve at the point P with coordinates (x, y) meets the x-axis at T. The point N is the foot of the perpendicular from P to the x-axis. The curve is such that, for all values of x, the gradient of the curve is positive and $TN = 2$.

(i) Show that the differential equation satisfied by x and y is $\frac{dy}{dx} = \frac{1}{2}y$. **[1]**

The point with coordinates (4, 3) lies on the curve.

(ii) Solve the differential equation to obtain the equation of the curve, expressing y in terms of x. **[5]**

2. M/J 18/P31/Q6

In a certain chemical reaction the amount, x grams, of a substance is decreasing. The differential equation relating x and t, the time in seconds since the reaction started, is

$$\frac{dx}{dt} = -kx\sqrt{t},$$

where k is a positive constant. It is given that $x = 100$ at the start of the reaction.

(i) Solve the differential equation, obtaining a relation between x, t and k. [5]

(ii) Given that $t = 25$ when $x = 80$, find the value of t when $x = 40$. [3]

3. M/J 18/P33/Q6

(i) Express $\frac{1}{4 - y^2}$ in partial fractions. [2]

(ii) The variables x and y satisfy the differential equation

$$x\frac{dy}{dx} = 4 - y^2,$$

and $y = 1$ when $x = 1$. Solve the differential equation, obtaining an expression for y in terms of x. [6]

4. O/N 17/P32/Q5

The variables x and y satisfy the differential equation

$$(x + 1)\frac{dy}{dx} = y(x + 2),$$

and it is given that $y = 2$ when $x = 1$. Solve the differential equation and obtain an expression for y in terms of x. [7]

5. O/N 17/P31/Q6, O/N 17/P33/Q6

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = 4\cos^2 y \tan x$$

for $0 \leqslant x < \frac{1}{2}\pi$, and $x = 0$ when $y = \frac{1}{4}\pi$. Solve this differential equation and find the value of x when $y = \frac{1}{3}\pi$. [8]

6. M/J 17/P32/Q5

In a certain chemical process a substance A reacts with and reduces a substance B. The masses of A and B at time t after the start of the process are x and y respectively. It is given that $\frac{dy}{dt} = -0.2xy$ and $x = \frac{10}{(1+t)^2}$. At the beginning of the process $y = 100$.

(i) Form a differential equation in y and t, and solve this differential equation. [6]

(ii) Find the exact value approached by the mass of B as t becomes large. State what happens to the mass of A as t becomes large. [2]

7. M/J 17/P33/Q6/iii

The equation $\cot x = 1 - x$ has one root in the interval $0 < x < \pi$, denoted by α.

(i) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

8. M/J 17/P33/Q8

In a certain chemical reaction, a compound A is formed from a compound B. The masses of A and B at time t after the start of the reaction are x and y respectively and the sum of the masses is equal to 50 throughout the reaction. At any time the rate of increase of the mass of A is proportional to the mass of B at that time.

(i) Explain why $\frac{dx}{dt} = k(50 - x)$, where k is a constant. [1]

It is given that $x = 0$ when $t = 0$, and $x = 25$ when $t = 10$.

(ii) Solve the differential equation in part **(i)** and express x in terms of t. [8]

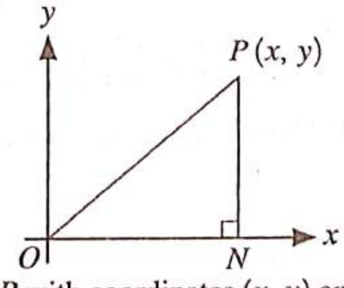
9. O/N 16/P32/Q10 , O/N 16/P31/Q10

A large field of area 4 km^2 is becoming infected with a soil disease. At time t years the area infected is $x\text{ km}^2$ and the rate of growth of the infected area is given by the differential equation $\frac{dx}{dt} = kx(4 - x)$, where k is a positive constant. It is given that when $t = 0$, $x = 0.4$ and that when $t = 2$, $x = 2$.

(i) Solve the differential equation and show that $k = \frac{1}{4}\ln 3$. [9]

(ii) Find the value of t when 90% of the area of the field is infected. [2]

10. O/N 16/P33/Q5



The diagram shows a variable point P with coordinates (x, y) and the point N which is the foot of the perpendicular from P to the x-axis. P moves on a curve such that, for all $x \geq 0$, the gradient of the curve is equal in value to the area of the triangle OPN, where O is the origin.

(i) State a differential equation satisfied by x and y. [1]

The point with coordinates $(0, 2)$ lies on the curve.

(ii) Solve the differential equation to obtain the equation of the curve, expressing y in terms of x. [5]

(iii) Sketch the curve. [1]

11. M/J 16/P32/Q6

The variables x and θ satisfy the differential equation

$$(3 + \cos 2\theta)\frac{dx}{d\theta} = x \sin 2\theta,$$

and it is given that $x = 3$ when $\theta = \frac{1}{4}\pi$.

(i) Solve the differential equation and obtain an expression for x in terms of θ. [7]

(ii) State the least value taken by x. [1]

12. M/J 16/P31/Q5

The curve with equation $y = \sin x \cos 2x$ has one stationary point in the interval $0 < x < \frac{1}{2}\pi$. Find the x-coordinate of this point, giving your answer correct to 3 significant figures. [6]

13. M/J 16/P31/Q6

(i) By sketching a suitable pair of graphs, show that the equation

$$5e^{-x} = \sqrt{x}$$

has one root. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{2}\ln\left(\frac{25}{x_n}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iii) Use this iterative formula, with initial value $x_1 = 1$, to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

14. M/J 16/P33/Q5

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{-2y}\tan^2 x,$$

for $0 \leqslant x < \frac{1}{2}\pi$, and it is given that $y = 0$ when $x = 0$. Solve the differential equation and calculate the value of y when $x = \frac{1}{4}\pi$. [8]

15. O/N 15/P32/Q8, O/N 15/P31/Q8

The variables x and θ satisfy the differential equation

$$\frac{dx}{d\theta} = (x+2)\sin^2 2\theta,$$

and it is given that $x = 0$ when $\theta = 0$. Solve the differential equation and calculate the value of x when $\theta = \frac{1}{4}\pi$, giving your answer correct to 3 significant figures. [9]

16. O/N 15/P33/Q10

Naturalists are managing a wildlife reserve to increase the number of plants of a rare species. The number of plants at time t years is denoted by N, where N is treated as a continuous variable.

(i) It is given that the rate of increase of N with respect to t is proportional to $(N - 150)$. Write down a differential equation relating N, t and a constant of proportionality. [1]

(ii) Initially, when $t = 0$, the number of plants was 650. It was noted that, at a time when there were 900 plants, the number of plants was increasing at a rate of 60 per year. Express N in terms of t. [7]

(iii) The naturalists had a target of increasing the number of plants from 650 to 2000 within 15 years. Will this target be met? [2]

17. M/J 15/P32/Q9

The number of organisms in a population at time t is denoted by x. Treating x as a continuous variable, the differential equation satisfied by x and t is

$$\frac{dx}{dt} = \frac{xe^{-t}}{k + e^{-t}},$$

where k is a positive constant.

(i) Given that $x = 10$ when $t = 0$, solve the differential equation, obtaining a relation between x, k and t. [6]

(ii) Given also that $x = 20$ when $t = 1$, show that $k = 1 - \frac{2}{e}$. [2]

(iii) Show that the number of organisms never reaches 48, however large t becomes. [2]

18. M/J 15/P31/Q7

Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = 4x(3y^2 + 10y + 3),$$

obtaining an expression for y in terms of x. [9]

19. M/J 15/P33/Q7

The number of micro-organisms in a population at time t is denoted by M. At any time the variation in M is assumed to satisfy the differential equation

$$\frac{dM}{dt} = k(\sqrt{M})\cos(0.02t),$$

where k is a constant and M is taken to be a continuous variable. It is given that when $t = 0$, $M = 100$.

(i) Solve the differential equation, obtaining a relation between M, k and t. [5]

(ii) Given also that $M = 196$ when $t = 50$, find the value of k. [2]

(iii) Obtain an expression for M in terms of t and find the least possible number of micro-organisms. [2]

20. O/N 14/P32/Q7, O/N 14/P31/Q7

In a certain country the government charges tax on each litre of petrol sold to motorists. The revenue per year is R million dollars when the rate of tax is x dollars per litre. The variation of R with x is modelled by the differential equation

$$\frac{dR}{dx} = R\left(\frac{1}{x} - 0.57\right),$$

where R and x are taken to be continuous variables. When $x = 0.5$, $R = 16.8$.

(i) Solve the differential equation and obtain an expression for R in terms of x. [6]

(ii) This model predicts that R cannot exceed a certain amount. Find this maximum value of R. [3]

21. O/N 14/P33/Q8

The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \tfrac{1}{5}xy^{\frac{1}{2}}\sin\left(\tfrac{1}{3}x\right).$$

(i) Find the general solution, giving y in terms of x. [6]

(ii) Given that $y = 100$ when $x = 0$, find the value of y when $x = 25$. [3]

22. M/J 14/P32/Q4

The parametric equations of a curve are

$$x = t - \tan t, \quad y = \ln(\cos t),$$

for $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$.

(i) Show that $\frac{dy}{dx} = \cot t$. [5]

(ii) Hence find the x-coordinate of the point on the curve at which the gradient is equal to 2. Give your answer correct to 3 significant figures. [2]

23. M/J 14/P32/Q9

The population of a country at time t years is N millions. At any time, N is assumed to increase at a rate proportional to the product of N and $(1 - 0.01N)$. When $t = 0$, $N = 20$ and $\frac{dN}{dt} = 0.32$.

(i) Treating N and t as continuous variables, show that they satisfy the differential equation

$$\frac{dN}{dt} = 0.02N(1 - 0.01N).$$ [1]

(ii) Solve the differential equation, obtaining an expression for t in terms of N. [8]

(iii) Find the time at which the population will be double its value at $t = 0$. [1]

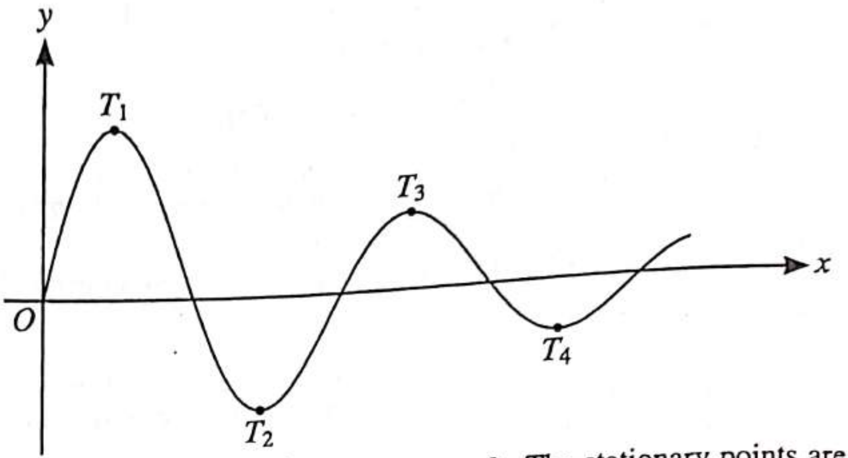
24. M/J 14/P31/Q4

The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \frac{6ye^{3x}}{2 + e^{3x}}.$$

Given that $y = 36$ when $x = 0$, find an expression for y in terms of x. [6]

25. M/J 14/P31/Q10



The diagram shows the curve $y = 10e^{-\frac{1}{2}x}\sin 4x$ for $x \geqslant 0$. The stationary points are labelled T_1, T_2, T_3, … as shown.

(i) Find the x-coordinates of T_1 and T_2, giving each x-coordinate correct to 3 decimal places. [6]

(ii) It is given that the x-coordinate of T_n is greater than 25. Find the least possible value of n. [4]

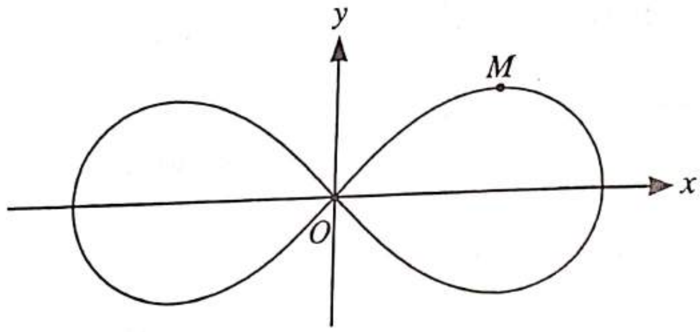
26. M/J 14/P33/Q5

The variables x and θ satisfy the differential equation

$$2\cos^2\theta \frac{dx}{d\theta} = \sqrt{(2x+1)},$$

and $x = 0$ when $\theta = \frac{1}{4}\pi$. Solve the differential equation and obtain an expression for x in terms of θ. [7]

27. M/J 14/P33/Q6

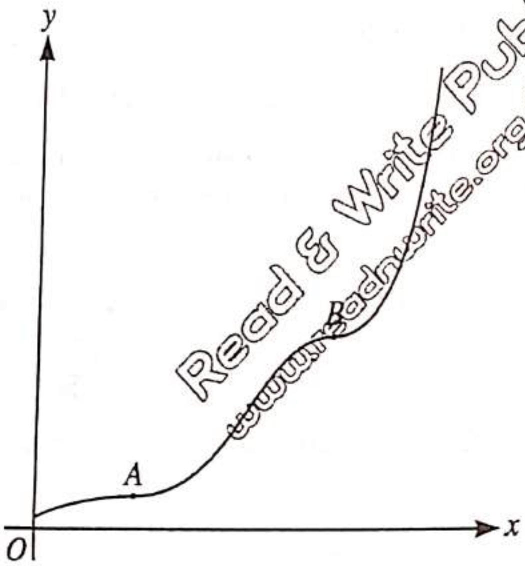


The diagram shows the curve $(x^2 + y^2)^2 = 2(x^2 - y^2)$ and one of its maximum points M. Find the coordinates of M. [7]

28. O/N 13/P33/Q4

A curve has equation $3e^{2x}y + e^xy^3 = 14$. Find the gradient of the curve at the point $(0, 2)$. [5]

29. O/N 13/P33/Q10



A particular solution of the differential equation

$$3y^2\frac{dy}{dx} = 4(y^3+1)\cos^2 x$$

is such that $y = 2$ when $x = 0$. The diagram shows a sketch of the graph of this solution for $0 \leqslant x \leqslant 2\pi$; the graph has stationary points at A and B. Find the y-coordinates of A and B, giving each coordinate correct to 1 decimal place. **[10]**

30. O/N 12/P32/Q9, O/N 12/P31/Q9

The complex number $1 + (\sqrt{2})\mathrm{i}$ is denoted by u. The polynomial $x^4 + x^2 + 2x + 6$ is denoted by $\mathrm{p}(x)$.

(i) Showing your working, verify that u is a root of the equation $\mathrm{p}(x) = 0$, and write down a second complex root of the equation. **[4]**

(ii) Find the other two roots of the equation $\mathrm{p}(x) = 0$. **[6]**

31. O/N 11/P32/Q4, O/N 11/P31/Q4

The variables x and θ are related by the differential equation

$$\sin 2\theta \frac{dx}{d\theta} = (x+1)\cos 2\theta,$$

where $0 < \theta < \frac{1}{2}\pi$. When $\theta = \frac{1}{12}\pi$, $x = 0$. Solve the differential equation, obtaining an expression for x in terms of θ, and simplifying your answer as far as possible. **[7]**

32. O/N 11/P33/Q4

During an experiment, the number of organisms present at time t days is denoted by N, where N is treated as a continuous variable. It is given that

$$\frac{dN}{dt} = 1.2e^{-0.02t}N^{0.5}.$$

When $t = 0$, the number of organisms present is 100.

(i) Find an expression for N in terms of t. **[6]**

(ii) State what happens to the number of organisms present after a long time. **[1]**

33. M/J 11/P32/Q6

A certain curve is such that its gradient at a point (x, y) is proportional to xy. At the point $(1, 2)$ the gradient is 4.

(i) By setting up and solving a differential equation, show that the equation of the curve is $y = 2e^{x^2-1}$. **[7]**

(ii) State the gradient of the curve at the point $(-1, 2)$ and sketch the curve. **[2]**

34. M/J 11/P31/Q10

The number of birds of a certain species in a forested region is recorded over several years. At time t years, the number of birds is N, where N is treated as a continuous variable. The variation in the number of birds is modelled by

$$\frac{dN}{dt} = \frac{N(1800-N)}{3600}.$$

It is given that $N = 300$ when $t = 0$.

(i) Find an expression for N in terms of t. **[9]**

(ii) According to the model, how many birds will there be after a long time? **[1]**

35. M/J 11/P33/Q9

In a chemical reaction, a compound X is formed from two compounds Y and Z. The masses in grams of X, Y and Z present at time t seconds after the start of the reaction are x, $10 - x$ and $20 - x$ respectively. At any time the rate of formation of X is proportional to the product of the masses of Y and Z present at the time. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 2$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.01(10 - x)(20 - x).$$ [1]

(ii) Solve this differential equation and obtain an expression for x in terms of t. [9]

(iii) State what happens to the value of x when t becomes large. [1]

36. O/N 10/P32/Q10 , O/N 10/P31/Q10

A certain substance is formed in a chemical reaction. The mass of substance formed t seconds after the start of the reaction is x grams. At any time the rate of formation of the substance is proportional to $(20 - x)$. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 1$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.05(20 - x).$$ [2]

(ii) Find, in any form, the solution of this differential equation. [5]

(iii) Find x when $t = 10$, giving your answer correct to 1 decimal place. [2]

(iv) State what happens to the value of x as t becomes very large. [1]

37. O/N 10/P33/Q9

A biologist is investigating the spread of a weed in a particular region. At time t weeks after the start of the investigation, the area covered by the weed is A m^2. The biologist claims that the rate of increase of A is proportional to $\sqrt{(2A - 5)}$.

(i) Write down a differential equation representing the biologist's claim. [1]

(ii) At the start of the investigation, the area covered by the weed was 7 m^2 and, 10 weeks later, the area covered was 27 m^2. Assuming that the biologist's claim is correct, find the area covered 20 weeks after the start of the investigation. [9]

38. M/J 10/P32/Q7

The variables x and t are related by the differential equation

$$e^{2t}\frac{dx}{dt} = \cos^2 x,$$

where $t \geqslant 0$. When $t = 0$, $x = 0$.

(i) Solve the differential equation, obtaining an expression for x in terms of t. [6]

(ii) State what happens to the value of x when t becomes very large. [1]

(iii) Explain why x increases as t increases. [1]

39. M/J 10/P31/Q5

Given that $y = 0$ when $x = 1$, solve the differential equation

$$xy\frac{dy}{dx} = y^2 + 4,$$

obtaining an expression for y^2 in terms of x. [6]

40. M/J 10/P33/Q4

Given that $x = 1$ when $t = 0$, solve the differential equation

$$\frac{dx}{dt} = \frac{1}{x} - \frac{x}{4},$$

obtaining an expression for x^2 in terms of t. [7]

41. O/N 09/P32/Q9

The temperature of a quantity of liquid at time t is θ. The liquid is cooling in an atmosphere whose temperature is constant and equal to A. The rate of decrease of θ is proportional to the temperature difference $(\theta - A)$. Thus θ and t satisfy the differential equation

$$\frac{d\theta}{dt} = -k(\theta - A),$$

where k is a positive constant.

(i) Find, in any form, the solution of this differential equation, given that $\theta = 4A$ when $t = 0$. [5]

(ii) Given also that $\theta = 3A$ when $t = 1$, show that $k = \ln \frac{3}{2}$. [1]

(iii) Find θ in terms of A when $t = 2$, expressing your answer in its simplest form. [3]

42. O/N 09/P31/Q10

In a model of the expansion of a sphere of radius r cm, it is assumed that, at time t seconds after the start, the rate of increase of the surface area of the sphere is proportional to its volume. When $t = 0$, $r = 5$ and $\frac{dr}{dt} = 2$.

(i) Show that r satisfies the differential equation

$$\frac{dr}{dt} = 0.08r^2.$$ [4]

[The surface area A and volume V of a sphere of radius r are given by the formulae $A = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$.]

(ii) Solve this differential equation, obtaining an expression for r in terms of t. [5]

(iii) Deduce from your answer to part **(ii)** the set of values that t can take, according to this model. [1]

43. M/J 09/P03/Q8

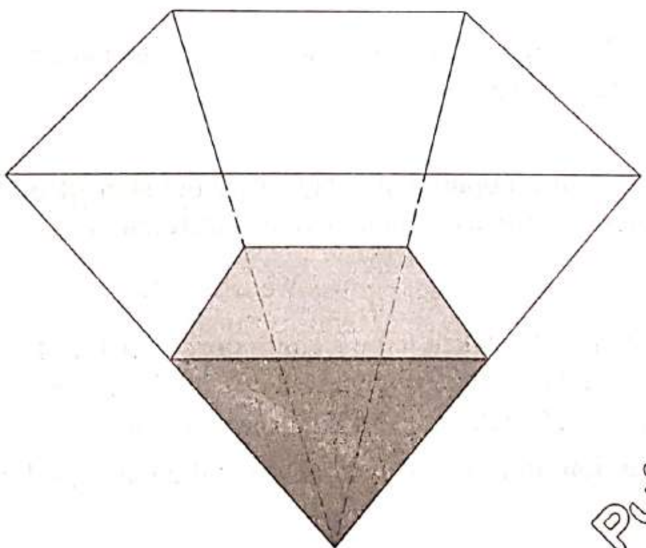
(i) Express $\frac{100}{x^2(10-x)}$ in partial fractions. [4]

(ii) Given that $x = 1$ when $t = 0$, solve the differential equation

$$\frac{dx}{dt} = \frac{1}{100}x^2(10-x),$$

obtaining an expression for t in terms of x. [6]

44. O/N 08/P03/Q8



An underground storage tank is being filled with liquid as shown in the diagram. Initially the tank is empty. At time t hours after filling begins, the volume of liquid is V m^3 and the depth of liquid is h m. It is given that $V = \frac{4}{3}h^3$.

The liquid is poured in at a rate of 20 m^3 per hour, but owing to leakage, liquid is lost at a rate proportional to h^2. When $h = 1$, $\frac{dh}{dt} = 4.95$.

(i) Show that h satisfies the differential equation

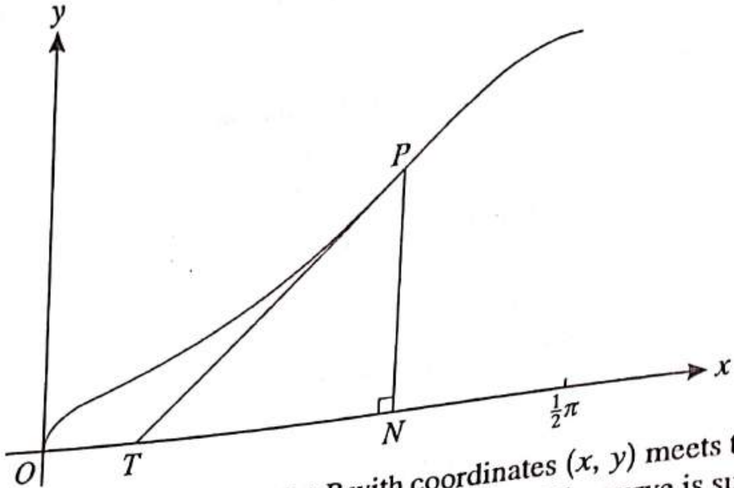
$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}.$$ [4]

[1]

(ii) Verify that $\dfrac{20h^2}{100-h^2} \equiv -20 + \dfrac{2000}{(10-h)(10+h)}$. [5]

(iii) Hence solve the differential equation in part **(i)**, obtaining an expression for t in terms of h.

45. M/J 08/P03/Q8



In the diagram the tangent to a curve at a general point P with coordinates (x, y) meets the x-axis at T. The point N on the x-axis is such that PN is perpendicular to the x-axis. The curve is such that, for all values of x in the interval $0 < x < \frac{1}{2}\pi$, the area of triangle PTN is equal to $\tan x$, where x is in radians.

(i) Using the fact that the gradient of the curve at P is $\dfrac{PN}{TN}$, show that

$$\frac{dy}{dx} = \tfrac{1}{2}y^2 \cot x.$$ [3]

(ii) Given that $y = 2$ when $x = \frac{1}{6}\pi$, solve this differential equation to find the equation of the curve, expressing y in terms of x. [6]

46. O/N 07/P03/Q7

The number of insects in a population t days after the start of observations is denoted by N. The variation in the number of insects is modelled by a differential equation of the form

$$\frac{dN}{dt} = kN\cos(0.02t),$$

where k is a constant and N is taken to be a continuous variable. It is given that $N = 125$ when $t = 0$.

(i) Solve the differential equation, obtaining a relation between N, k and t. [5]

(ii) Given also that $N = 166$ when $t = 30$, find the value of k. [2]

(iii) Obtain an expression for N in terms of t, and find the least value of N predicted by this model. [3]

47. M/J 07/P03/Q10

A model for the height, h metres, of a certain type of tree at time t years after being planted assumes that, while the tree is growing, the rate of increase in height is proportional to $(9-h)^{\frac{1}{3}}$. It is given that, when $t = 0$, $h = 1$ and $\dfrac{dh}{dt} = 0.2$.

(i) Show that h and t satisfy the differential equation

$$\frac{dh}{dt} = 0.1(9-h)^{\frac{1}{3}}.$$ [2]

(ii) Solve this differential equation, and obtain an expression for h in terms of t. [7]

(iii) Find the maximum height of the tree and the time taken to reach this height after planting. [2]

(iv) Calculate the time taken to reach half the maximum height. [1]

48. O/N 06/P03/Q4

Given that $y = 2$ when $x = 0$, solve the differential equation

$$y\frac{dy}{dx} = 1 + y^2,$$

obtaining an expression for y^2 in terms of x. [6]

49. M/J 06/P03/Q5

In a certain industrial process, a substance is being produced in a container. The mass of the substance in the container t minutes after the start of the process is x grams. At any time, the rate of formation of the substance is proportional to its mass. Also, throughout the process, the substance is removed from the container at a constant rate of 25 grams per minute. When $t = 0$, $x = 1000$ and $\frac{dx}{dt} = 75$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.1(x - 250).$$ [2]

(ii) Solve this differential equation, obtaining an expression for x in terms of t. [6]

50. O/N 05/P03/Q8

In a certain chemical reaction the amount, x grams, of a substance present is decreasing. The rate of decrease of x is proportional to the product of x and the time, t seconds, since the start of the reaction. Thus x and t satisfy the differential equation

$$\frac{dx}{dt} = -kxt,$$

where k is a positive constant. At the start of the reaction, when $t = 0$, $x = 100$.

(i) Solve this differential equation, obtaining a relation between x, k and t. [5]

(ii) 20 seconds after the start of the reaction the amount of substance present is 90 grams. Find the time after the start of the reaction at which the amount of substance present is 50 grams. [3]

51. M/J 05/P03/Q8

(i) Using partial fractions, find

$$\int \frac{1}{y(4-y)}\,dy.$$ [4]

(ii) Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = y(4 - y),$$

obtaining an expression for y in terms of x. [4]

(iii) State what happens to the value of y if x becomes very large and positive. [1]

52. O/N 04/P03/Q10

A rectangular reservoir has a horizontal base of area $1000\,m^2$. At time $t = 0$, it is empty and water begins to flow into it at a constant rate of $30\,m^3\,s^{-1}$. At the same time, water begins to flow out at a rate proportional to $\sqrt{h}$, where h m is the depth of the water at time t s. When $h = 1$, $\frac{dh}{dt} = 0.02$.

(i) Show that h satisfies the differential equation

$$\frac{dh}{dt} = 0.01(3 - \sqrt{h}).$$ [3]

It is given that, after making the substitution $x = 3 - \sqrt{h}$, the equation in part (i) becomes

$$(x - 3)\frac{dx}{dt} = 0.005x.$$

(ii) Using the fact that $x = 3$ when $t = 0$, solve this differential equation, obtaining an expression for t in terms of x. [5]

(iii) Find the time at which the depth of water reaches 4 m. [2]

53. M/J 04/P03/Q6

Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = \frac{y^3 + 1}{y^2},$$ [6]

obtaining an expression for y in terms of x.

54. O/N 03/P03/Q9

Compressed air is escaping from a container. The pressure of the air in the container at time t is P, and the constant atmospheric pressure of the air outside the container is A. The rate of decrease of P is proportional to the square root of the pressure difference $(P - A)$. Thus the differential equation connecting P and t is

$$\frac{dP}{dt} = -k\sqrt{(P - A)},$$

where k is a positive constant.

(i) Find, in any form, the general solution of this differential equation. [3]

(ii) Given that $P = 5A$ when $t = 0$, and that $P = 2A$ when $t = 2$, show that $k = \sqrt{A}$. [4]

(iii) Find the value of t when $P = A$. [2]

(iv) Obtain an expression for P in terms of A and t. [2]

55. M/J 03/P03/Q7

In a chemical reaction a compound X is formed from a compound Y. The masses in grams of X and Y present at time t seconds after the start of the reaction are x and y respectively. The sum of the two masses is equal to 100 grams throughout the reaction. At any time, the rate of formation of X is proportional to the mass of Y at that time. When $t = 0$, $x = 5$ and $\frac{dx}{dt} = 1.9$.

(i) Show that x satisfies the differential equation

$$\frac{dx}{dt} = 0.02(100 - x).$$ [2]

(ii) Solve this differential equation, obtaining an expression for x in terms of t. [6]

(iii) State what happens to the value of x as t becomes very large. [1]

56. O/N 02/P03/Q9

In an experiment to study the spread of a soil disease, an area of 10 m^2 of soil was exposed to infection. In a simple model, it is assumed that the infected area grows at a rate which is proportional to the product of the infected area and the uninfected area. Initially, 5 m^2 was infected and the rate of growth of the infected area was 0.1 m^2 per day. At time t days after the start of the experiment, an area a m^2 is infected and an area (10 – a)m^2 is uninfected.

(i) Show that $\frac{da}{dt} = 0.004a(10 - a)$. [2]

(ii) By first expressing $\frac{1}{a(10-a)}$ in partial fractions, solve this differential equation, obtaining an expression for t in terms of a. [6]

(iii) Find the time taken for 90% of the soil area to become infected, according to this model. [2]

57. M/J 02/P03/Q7

In a certain chemical process a substance is being formed, and t minutes after the start of the process there are m grams of the substance present. In the process the rate of increase of m is proportional to $(50-m)^2$. When $t=0$, $m=0$ and $\frac{dm}{dt}=5$.

(i) Show that m satisfies the differential equation **[2]**

$$\frac{dm}{dt}=0.002(50-m)^2$$

(ii) Solve the differential equation, and show that the solution can be expressed in the form **[5]**

$$m=50-\frac{500}{t+10}.$$

(iii) Calculate the mass of the substance when $t=10$, and find the time taken for the mass to increase from 0 to 45 grams. **[2]**

(iv) State what happens to the mass of the substance as t becomes very large. **[1]**

Answers Section

1. M/J 18/P32/Q3 1

(i) Fully justify the given statement

(ii) Separate variables and attempt integration of at least one side

Obtain terms $\ln y$ and $\frac{1}{2}x$

Use $x = 4, y = 3$ to evaluate a constant or as limits in a solution with terms $a \ln y$ and bx, where $ab \neq 0$ 5

Obtain correct solution in any form

Obtain answer $y = 3e^{\frac{1}{2}x-2}$, or equivalent

2. M/J 18/P31/Q6

(i) Separate variables correctly and integrate at least one side

Obtain term $\ln x$

Obtain term $-\frac{2}{3}kt\sqrt{t}$, or equivalent

Evaluate a constant, or use limits $x = 100$ and $t = 0$, in a solution containing terms $a \ln x$ and $bt\sqrt{t}$ 5

Obtain correct solution in any form, e.g. $\ln x = -\frac{2}{3}kt\sqrt{t} + \ln 100$

(ii) Substitute $x = 80$ and $t = 25$ to form equation in k 3

Substitute $x = 40$ and eliminate k

Obtain answer $t = 64.1$

3. M/J 18/P33/Q6

(i) Carry out relevant method to find A and B such that

$$\frac{1}{4-y^2} \equiv \frac{A}{2+y} + \frac{B}{2-y}$$ 2

Obtain $A = B = \frac{1}{4}$

(ii) Separate variables correctly and integrate at least one side to obtain one of the terms $a \ln x$, $b \ln(2+y)$ or $c \ln(2-y)$

Obtain term $\ln x$

Integrate and obtain terms $\frac{1}{4}\ln(2+y) - \frac{1}{4}\ln(2-y)$

Use $x = 1$ and $y = 1$ to evaluate a constant, or as limits, in a solution containing at least two terms of the form $a \ln x$, $b \ln(2+y)$ and $c \ln(2-y)$

Obtain a correct solution in any form, e.g.

$$\ln x = \frac{1}{4}\ln(2+y) - \frac{1}{4}\ln(2-y) - \frac{1}{4}\ln 3$$ 6

Rearrange as $\frac{2(3x^4-1)}{(3x^4+1)}$, or equivalent

4. O/N 17/P32/Q5

Separate variables and obtain $\int \frac{1}{y}\,dy = \int \frac{x+2}{x+1}\,dx$

Obtain term $\ln y$ 7

Use an appropriate method to integrate $(x+2)/(x+1)$

Obtain integral $x+\ln(x+1)$, or equivalent, e.g. $\ln(x+1)+x+1$
Use $x=1$ and $y=2$ to evaluate a constant, or as limits
Obtain correct solution in x and y in any form e.g. $\ln y = x+\ln(x+1)-1$
Obtain answer $y=(x+1)e^{x-1}$

5. **O/N 17/P31/Q6, O/N 17/P33/Q6**
Separate variables correctly and attempt integration of one side
Obtain term $\tan y$, or equivalent
Obtain term of the form $k\ln\cos x$, or equivalent
Obtain term $-4\ln\cos x$, or equivalent
Use $x=0$ and $y=\frac{1}{4}\pi$ in solution containing $a\tan y$ and $b\ln\cos x$ to evaluate a constant, or as limits
Obtain correct solution in any form, e.g. $\tan y = 4\ln\sec x+1$
Substitute $y=\frac{1}{3}\pi$ in solution containing terms $a\tan y$ and $b\ln\cos x$, and use correct method to find x 8
Obtain answer $x=0.587$

6. **M/J 17/P32/Q5**
(i) State $\frac{dy}{dt}=-\frac{2y}{(1+t)^2}$, or equivalent
Separate variables correctly and attempt integration of one side
Obtain term $\ln y$, or equivalent
Obtain term $\frac{2}{(1+t)}$, or equivalent
Use $y=100$ and $t=0$ to evaluate a constant, or as limits in an expression containing terms of the form $a\ln y$ and $\frac{b}{1+t}$
Obtain correct solution in any form, e.g. $\ln y = \frac{2}{1+t}-2+\ln 100$ 6
(ii) State that the mass of B approaches $\frac{100}{e^2}$, or exact equivalent
State or imply that the mass of A tends to zero 2

7. **M/J 17/P33/Q6/iii**
(i) Use the iterative formula correctly at least once
Obtain final answer 2.576 only
Show sufficient iterations to 5 d.p. to justify 2.576 to 3 d.p., or show there is a sign change in the interval (2.5755, 2.5765) 3

8. **M/J 17/P33/Q8**
(i) Justify the given differential equation 1
(ii) Separate variables correctly and attempt to integrate one side
Obtain term kt, or equivalent
Obtain term $-\ln(50-x)$, or equivalent
Evaluate a constant, or use limits $x=0$, $t=0$ in a solution containing terms $a\ln(50-x)$ and bt
Obtain solution $-\ln(50-x)=kt-\ln 50$, or equivalent
Use $x=25$, $t=10$ to determine k
Obtain correct solution in any form, e.g. $\ln 50-\ln(50-x)=\frac{1}{10}(\ln 2)t$
Obtain answer $x=50(1-\exp(-0.0693t))$, or equivalent 8

9. O/N 16/P32/Q10 , O/N 16/P31/Q10

(i) Separate variables correctly and integrate at least one side
Integrate and obtain term kt, or equivalent
Carry out a relevant method to obtain A and B such that $\frac{1}{x(4-x)} \equiv \frac{A}{x} + \frac{B}{4-x}$, or equivalent

Obtain $A = B = \frac{1}{4}$, or equivalent

Integrate and obtain terms $\frac{1}{4}\ln x - \frac{1}{4}\ln(4-x)$, or equivalent

EITHER: Use a pair of limits in an expression containing $p\ln x$, $q\ln(4-x)$ and rt and evaluate a constant
Obtain correct answer in any form, e.g. $\ln x - \ln(4-x) = 4kt - \ln 9$,

or $\ln\left(\frac{x}{4-x}\right) = 4kt - 8k$

Use a second pair of limits and determine k
Obtain the given exact answer correctly

OR: Use both pairs of limits in a definite integral
Obtain the given exact answer correctly
Substitute k and either pair of limits in an expression containing $p\ln x$, $q\ln(4-x)$ and rt and evaluate a constant **[9]**

Obtain $\ln\frac{x}{4-x} = t\ln 3 - \ln 9$ or equivalent **[2]**

(ii) Substitute $x = 3.6$ and solve for t
Obtain answer $t = 4$ **[1]**

10. O/N 16/P33/Q5

(i) State equation $\frac{dy}{dx} = \frac{1}{2}xy$

(ii) Separate variables correctly and attempts to integrate one side of equation
Obtain terms of the form $a\ln y$ and bx^2
Use $x = 0$ and $y = 2$ to evaluate a constant, or as limits, in expression containing a ln y or bx^2
Obtain correct solution in any form, e.g. $\ln y = \frac{1}{4}x^2 + \ln 2$ **[5]**

Obtain correct expression for y, e.g. $y = 2e^{\frac{1}{4}x^2}$ **[1]**

(iii) Show correct sketch for $x \geqslant 0$. Needs through (0, 2) and rapidly increasing positive gradient.

11. M/J 16/P32/Q6

(i) Separate variables correctly and attempt integration of at least one side
Obtain term $\ln x$
Obtain term of the form $k\ln(3+\cos 2\theta)$, or equivalent
Obtain term $-\frac{1}{2}\ln(3+\cos 2\theta)$, or equivalent
Use $x = 3$, $\theta = \frac{1}{4}\pi$ to evaluate a constant or as limits in a solution with terms $a\ln x$ and $b\ln(3+\cos 2\theta)$, where $ab \neq 0$
State correct solution in any form, e.g. $\ln x = -\frac{1}{2}\ln(3+\cos 2\theta) + \frac{3}{2}\ln 3$ **[7]**

Rearrange in a correct form, e.g. $x = \sqrt{\left(\frac{27}{3+\cos 2\theta}\right)}$

(ii) State answer $x = 3\sqrt{3}/2$, or exact equivalent (accept decimal answer in [2.59, 2.60]) **[1]**

12. **M/J 16/P31/Q5**
Use product rule
Obtain correct derivative in any form, e.g. $\cos x \cos 2x - 2\sin x \sin 2x$
Equate derivative to zero and use double angle formulae
Remove factor of $\cos x$ and reduce equation to one in a single trig function
Obtain $6\sin^2 x = 1$, $6\cos^2 x = 5$ or $5\tan^2 x = 1$
Solve and obtain $x = 0.421$
[Alternative: Use double angle formula M1. Use chain rule to differentiate M1. Obtain correct derivative
e.g. $\cos\theta - 6\sin^2\theta\cos\theta$ A1, then as above.] **[6]**

13. **M/J 16/P31/Q6**
(i) Make recognizable sketch of a relevant graph **[2]**
Sketch the other relevant graph and justify the given statement
(ii) State $x = \frac{1}{2}\ln(25/x)$ **[2]**
Rearrange this in the form $5e^{-x} = \sqrt{x}$
(iii) Use the iterative formula correctly at least once
Obtain final answer 1.43
Show sufficient iterations to 4 d.p. to justify 1.43 to 2 d.p., or show there is a sign change in the **[3]**
interval (1.425, 1.435)

14. **M/J 16/P33/Q5**
Separate variables and make reasonable attempt at integration of either integral
Obtain term $\frac{1}{2}e^{2y}$
Use Pythagoras
Obtain terms $\tan x - x$
Evaluate a constant or use $x = 0, y = 0$ as limits in a solution containing terms
$ae^{\pm 2y}$ and $b\tan x$, ($ab \neq 0$)
Obtain correct solution in any form, e.g. $\frac{1}{2}e^{2y} = \tan x - x + \frac{1}{2}$
Set $x = \frac{1}{4}\pi$ and use correct method to solve an equation of the form $e^{\pm 2y} = a$ or $e^{\pm y} = a$, where
$a > 0$ **[8]**
Obtain answer $y = 0.179$

15. **O/N 15/P32/Q8, O/N 15/P31/Q8**
Separate variables and integrate one side
Obtain term $\ln(x+2)$
Use $\cos 2A$ formula to express $\sin^2 2\theta$ in the form $a + b\cos 4\theta$
Obtain correct form $(1 - \cos 4\theta)/2$, or equivalent
Integrate and obtain term $\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta$, or equivalent
Evaluate a constant, or use $\theta = 0, x = 0$ as limits in a solution containing terms
$c\ln(x+2), d\sin(4\theta), e\theta$
Obtain correct solution in any form, e.g. $\ln(x+2) = \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta + \ln 2$
Use correct method for solving an equation of the form $\ln(x+2) = f$ **[9]**
Obtain answer $x = 0.962$

16. O/N 15/P33/Q10

(i) State $\frac{dN}{dt} = k(N - 150)$ **[1]**

(ii) Substitute $\frac{dN}{dt} = 60$ and $N = 900$ to find value of k

Obtain $k = 0.08$

Separate variables and obtain general solution involving $\ln(N - 150)$

Obtain $\ln(N - 150) = 0.08t + c$ (following their k) or $\ln(N - 150) = kt + c$

Substitute $t = 0$ and $N = 650$ to find c

Obtain $\ln(N - 150) = 0.08t + \ln 500$ or equivalent

Obtain $N = 500e^{0.08t} + 150$ **[7]**

(iii) Either Substitute $t = 15$ to find N or solve for t with $N = 2000$

Obtain Either $N = 1810$ or $t = 16.4$ and conclude target not met **[2]**

17. M/J 15/P32/Q9

(i) Separate variables correctly and attempt integration of one side

Obtain term $\ln x$

Obtain term of the form $a\ln(k + e^{-t})$

Obtain term $-\ln(k + e^{-t})$

Evaluate a constant or use limits $x = 10$, $t = 0$ in a solution containing terms $a\ln(k + e^{-t})$ and $b\ln x$

Obtain correct solution in any form, e.g. $\ln x - \ln 10 = -\ln(k + e^{-t}) + \ln(k + 1)$ **[6]**

(ii) Substitute $x = 20$, $t = 1$ and solve for k

Obtain the given answer **[2]**

(iii) Using $e^{-t} \to 0$ and the given value of k, find the limiting value of x

Justify the given answer **[2]**

18. M/J 15/P31/Q7

Separate variables and factorise to obtain $\frac{dy}{(3y+1)(y+3)} = 4x\,dx$ or equivalent

State or imply the form $\frac{A}{3y+1} + \frac{B}{y+3}$ and use a relevant method to find A or B

Obtain $A = \frac{3}{8}$ and $B = -\frac{1}{8}$

Integrate to obtain form $k_1\ln(3y+1) + k_2\ln(y+3)$

Obtain correct $\frac{1}{8}\ln(3y+1) - \frac{1}{8}\ln(y = 3) = 2x^2$ or equivalent

Substitute $x = 0$ and $y = 1$ in equation of form $k_1\ln(3y+1) + k_2\ln(y+3) = k_3x^2$ to find a value of c

Obtain $c = 0$

Use correct process to obtain equation without natural logarithm present

Obtain $y = \frac{3e^{16x^2} - 1}{3 - e^{16x^2}}$ or equivalent **[9]**

19. M/J 15/P33/Q7

(i) Separate variables correctly and integrate one side

Obtain term $2\sqrt{M}$, or equivalent

Obtain term $50k\sin(0.02t)$, or equivalent

Evaluate a constant of integration, or use limits $M = 100$, $t = 0$ in a solution with terms of the form $a\sqrt{M}$ and $b\sin(0.02t)$

Obtain correct solution in any form, e.g. $2\sqrt{M} = 50k\sin(0.02t) + 20$ **5**

(ii) Use values $M=196$, $t=50$ and calculate k
Obtain answer $k=0.190$ **2**

(iii) State an expression for M in terms of t, e.g. $M=(4.75\sin(0.02t)+10)^2$
State that the least possible number of micro-organisms is 28 or 27.5 or 27.6 (27.5625) **2**

20. O/N 14/P32/Q7, O/N 14/P31/Q7

(i) Separate variables correctly and attempt to integrate at least one side
Obtain term $\ln R$
Obtain $\ln x - 0.57x$
Evaluate a constant or use limits $x=0.5$, $R=16.8$, in a solution containing terms of the form $a\ln R$ and $b\ln x$
Obtain correct solution in any form
Obtain a correct expression for R, e.g. $R = xe^{(3.80-0.57x)}$, $R=44.7xe^{-0.57x}$ or $R=33.6xe^{(0.285-0.57x)}$ **[6]**

(ii) Equate $\frac{dR}{dx}$ to zero and solve for x
State or imply $x=0.57^{-1}$, or equivalent, e.g. 1.75
Obtain $R=28.8$ (allow 28.9) **[3]**

21. O/N 14/P33/Q8

(i) Sensibly separate variables and attempt integration of at least one side
Obtain $2y^{\frac{1}{2}} = \dots$ or equivalent
Correct integration by parts of $x\sin\frac{1}{3}x$ as far as $ax\cos\frac{1}{3}x \pm \int b\cos\frac{1}{3}x\,dx$
Obtain $-3x\cos\frac{1}{3}x + \int 3\cos\frac{1}{3}x\,dx$ or equivalent
Obtain $-3x\cos\frac{1}{3}x + 9\sin\frac{1}{3}x$ or equivalent
Obtain $y=\left(-\frac{3}{10}x\cos\frac{1}{3}x+\frac{9}{10}\sin\frac{1}{3}x+c\right)^2$ or equivalent **[6]**

(ii) Use $x=0$ and $y=100$ to find constant
Substitute 25 and calculate value of y
Obtain 203 **[3]**

22. M/J 14/P32/Q4

(i) State $\frac{dx}{dt}=1-\sec^2 t$, or equivalent **5**
Use chain rule
Obtain $\frac{dy}{dt}=-\frac{\sin t}{\cos t}$, or equivalent
Use $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$
Obtain the given answer correctly.

(ii) State or imply $t=\tan^{-1}(\frac{1}{2})$ **2**
Obtain answer $x=-0.0364$

23. M/J 14/P32/Q9

(i) State or imply $\frac{dN}{dt}=kN(1-0.01N)$ and obtain the given answer $k=0.02$ **1**

(ii) Separate variables and attempt integration of at least one side
Integrate and obtain term $0.02t$, or equivalent
Carry out a relevant method to obtain A or B such that $\frac{1}{N(1-0.01N)} \equiv \frac{A}{N} + \frac{B}{1-0.01N}$, or equivalent
Obtain $A = 1$ and $B = 0.01$, or equivalent
Integrate and obtain terms $\ln N - \ln(1-0.01N)$, or equivalent
Evaluate a constant or use limits $t = 0$, $N = 20$ in a solution with terms $a \ln N$ and $b\ln(1-0.01N)$, $ab \neq 0$ M1(dep*)
Obtain correct answer in any form, e.g. $\ln N - \ln(1-0.01N) = 0.02t + \ln 25$
Rearrange and obtain $t = 50\ln(4N/(100-N))$, or equivalent 8

(iii) Substitute $N = 40$ and obtain $t = 49.0$ 1

24. M/J 14/P31/Q4

Separate variables correctly and recognisable attempt at integration of at least one side
Obtain $\ln y$, or equivalent
Obtain $k\ln\left(2+e^{3x}\right)$
Use $y(0) = 36$ to find constant in $y = A\left(2+e^{3x}\right)^k$ or $\ln y = k\ln\left(2+e^{3x}\right)+c$ or equivalent
Obtain equation correctly without logarithms from $\ln y = \ln\left(A\left(2+e^{3x}\right)^k\right)$ [6]
Obtain $y = 4\left(2+e^{3x}\right)^2$

25. M/J 14/P31/Q10

(i) Use of product or quotient rule
Obtain $-5e^{-\frac{1}{2}x}\sin 4x + 40e^{-\frac{1}{2}x}\cos 4x$
Equate $\frac{dy}{dx}$ to zero and obtain $\tan 4z = k$ or $R\cos(4x \pm \alpha)$
Obtain $\tan 4x = 8$ or $\sqrt{65}\cos\left(4x \pm \tan^{-1}\frac{1}{8}\right)$
Obtain 0.362 or 20.7°
Obtain 1.147 or 65.7° [6]

(ii) State or imply that x-coordinates of T_n are increasing by $\frac{1}{4}\pi$ or 45°
Attempt solution of inequality (or equation) of form $x_1 + (n-1)k\pi$. 25
Obtain $n > \frac{4}{\pi}(25-0.362)+1$, following through on their value of x_1 [4]
$n = 33$

26. M/J 14/P33/Q5

Separate variables correctly and attempt integration of at least one side
Obtain term in the form $a\sqrt{(2x+1)}$
Express $1/(\cos^2\theta)$ as $\sec^2\theta$
Obtain term of the form $k\tan\theta$
Evaluate a constant, or use limits $x = 0, \theta = \frac{1}{4}\pi$ in a solution with terms $a\sqrt{(2x+1)}$ and $k\tan\theta$, $ak \neq 0$
Obtain correct solution in any form, e.g. $\sqrt{(2x+1)} = \frac{1}{2}\tan\theta + \frac{1}{2}$ 7
Rearrange and obtain $x = \frac{1}{8}(\tan\theta+1)^2 - \frac{1}{2}$, or equivalent

27. M/J 14/P33/Q6

Obtain correct derivative of RHS in any form
Obtain correct derivative of LHS in any form
Set $\frac{dy}{dx}$ equal to zero and obtain a horizontal equation
Obtain a correct equation, e.g. $x^2 + y^2 = 1$, from correct work
By substitution in the curve equation, or otherwise, obtain an equation in x^2 or y^2
Obtain $x = \frac{1}{2}\sqrt{3}$ **7**
Obtain $y = \frac{1}{2}$

28. O/N 13/P33/Q4

Differentiate y^3 to obtain $3y^2 \frac{dy}{dx}$
Use correct product rule at least once
Obtain $6e^{2x}y + 3e^{2x}\frac{dy}{dx} + e^x y^3 + 3e^x y^2 \frac{dy}{dx}$ as derivative of LHS
Equate derivative of LHS to zero, substitute $x = 0$ and $y = 2$ and find value of $\frac{dy}{dx}$ **[5]**
Obtain $-\frac{4}{3}$ or equivalent as **final answer**

29. O/N 13/P33/Q10

Use $2\cos^2 x = 1 + \cos 2x$ or equivalent
Separate variables and integrate at least one side
Obtain $\ln\left(y^3 + 1\right) = \ldots$ or equivalent
Obtain $\ldots = 2x + \sin 2x$ or equivalent
Use $x = 0$, $y = 2$ to find constant of integration (or as limits) in an expression containing at least two terms of the form $a\ln\left(y^3 + 1\right)$, bx or $c \sin 2x$
Obtain $\ln\left(y^3 + 1\right) = 2x + \sin 2x + \ln 9$ or equivalent e.g. implied by correct constant
Identify at least one of $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$ as x-coordinate at stationary point
Use correct process to find y-coordinate for at least one x-coordinate
Obtain 5.9 **[10]**
Obtain 48.1

30. O/N 12/P32/Q9 , O/N 12/P31/Q9

(i) *EITHER* Substitute $x = 1 + \sqrt{2}\,i$ and attempt the expansions of the x^2 and x^4 terms
Use $i^2 = -1$ correctly at least once
Complete the verification
State second root $1 - \sqrt{2}\,i$

OR 1 State second root $1 - \sqrt{2}\,i$
Carry out a complete method for finding a quadratic factor with zeros $1 \pm \sqrt{2}\,i$
Obtain $x^2 - 2x + 3$, or equivalent
Show that the division of p(x) by $x^2 - 2x + 3$ gives zero remainder and complete the verification

OR 2 Substitute $x = 1 + \sqrt{2}\,i$ and use correct method to express x^2 and x^4 in polar form
Obtain x^2 and x^4 in any correct polar form (allow decimals here)
Complete an exact verification
State second root $1 - \sqrt{2}\,i$, or its polar equivalent (allow decimals here) **[4]**

37. O/N 10/P33/Q9

(i) State $\frac{dA}{dt} = k\sqrt{2A-5}$ [1]

(ii) Separate variables correctly and attempt integration of each side
Obtain $(2A-5)^{\frac{1}{2}} = \ldots$ or equivalent
Obtain $= kt$ or equivalent
Use $t = 0$ and $A = 7$ to find value of arbitrary constant
Obtain $C = 3$ or equivalent
Use $t = 10$ and $A = 27$ to find k [9]
Obtain $k = 0.4$ or equivalent
Substitute $t = 20$ and values for C and k to find value of A
Obtain 63

38. M/J 10/P32/Q7

(i) Separate variables correctly and attempt integration of both sides
Obtain term $\tan x$
Obtain term $-\frac{1}{2}e^{-2t}$
Evaluate a constant or use limits $x = 0$, $t = 0$ in a solution containing terms $a\tan x$ and be^{-2t}
Obtain correct solution in any form, e.g. $\tan x = \frac{1}{2} - \frac{1}{2}e^{-2t}$ [6]
Rearrange as $x = \tan^{-1}(\frac{1}{2} - \frac{1}{2}e^{-2t})$, or equivalent [1]

(ii) State that x approaches $\tan^{-1}(\frac{1}{2})$

(iii) State that $1 - e^{-2t}$ increases and so does the inverse tangent, or state that $e^{-2t}\cos^2 x$ is positive [1]

39. M/J 10/P31/Q5

Separate variables correctly
Integrate and obtain term $\ln x$
Integrate and obtain term $\frac{1}{2}\ln(y^2 + 4)$
Evaluate a constant or use limits $y = 0$, $x = 1$ in a solution containing $a\ln x$ and $b\ln(y^2 + 4)$
Obtain correct solution in any form, e.g. $\frac{1}{2}\ln(y^2 + 4) = \ln x + \frac{1}{2}\ln 4$ [6]
Rearrange as $y^2 = 4(x^2 - 1)$, or equivalent

40. M/J 10/P33/Q4

Separate variables correctly
Obtain term $k\ln(4 - x^2)$, or terms $k_1\ln(2 - x) + k_2\ln(2 + x)$
Obtain term $-2\ln(4 - x^2)$, or $-2\ln(2 - x) - 2\ln(2 + x)$, or equivalent
Obtain term t, or equivalent
Evaluate a constant or use limits $x = 1$, $t = 0$ in a solution containing terms $a\ln(4 - x^2)$ and bt
or terms $c\ln(2 - x)$, $d\ln(2 + x)$ and bt
Obtain correct solution in any form, e.g. $-2\ln(4 - x^2) = t - 2\ln 3$ [7]
Rearrange and obtain $x^2 = 4 - 3\exp(-\frac{1}{2}t)$, or equivalent (allow use of $2\ln 3 = 2.20$)

41. O/N 09/P32/Q9

(i) Separate variables correctly
Integrate and obtain term $\ln(\theta - A)$, or equivalent
Integrate and obtain term $-kt$, or equivalent
Use $\theta = 4A$, $t = 0$ to determine a constant, or as limits [5]
Obtain correct answer in any form, e.g. $\ln(\theta - A) = -kt + \ln 3A$, with no errors seen [1]

(ii) Substitute $\theta = 3A$, $t = 1$ and justify the given statement

(iii) Substitute $t = 2$ and solve for θ in terms of A
Remove logarithms
Obtain answer $\theta = \frac{7}{3}A$, or equivalent, with no errors seen [3]
[The M marks are only available if the solution to part **(i)** contains terms $a\ln(\theta - A)$ and bt.]

42. O/N 09/P31/Q10

(i) State or imply $\frac{dA}{dt} = kV$

Obtain equation in r and $\frac{dr}{dt}$, e.g. $8\pi r\frac{dr}{dt} = k\frac{4}{3}\pi r^3$

Use $\frac{dr}{dt} = 2, r = 5$ to evaluate k

Obtain given answer **[4]**

(ii) Separate variables correctly and integrate both sides

Obtain terms $-\frac{1}{r}$ and $0.08t$, or equivalent

Evaluate a constant or use limits $t = 0, r = 5$ with a solution containing terms of the form $\frac{a}{r}$ and bt

Obtain solution $r = \frac{5}{(1-0.4t)}$, or equivalent **[5]**

(iii) State the set of values $0 \leq t < 2.5$, or equivalent **[1]**

[Allow $t < 2.5$ and $0 < t < 2.5$ to earn B1.]

43. M/J 09/P03/Q8

(i) State or imply the form $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{10-x}$

Use any relevant method to determine a constant

Obtain one of the values $A = 1, B = 10, C = 1$

Obtain the remaining two values

[The form $\frac{Dx+E}{x^2} + \frac{C}{10-x}$ is acceptable and leads to $D = 1, E = 10, C = 1$] **4**

(ii) Separate variables and attempt integration of both sides

Obtain terms $\ln x$, $-10/x$, $-\ln(10-x)$, or equivalent

Evaluate a constant or use limits $x = 1, t = 0$ with a solution containing 3 of the terms $k\ln x$, l/x, $m\ln(10-x)$ and t, or equivalent

Obtain any correct expression for t, e.g. $t = \ln\left(\frac{9x}{10-x}\right) - \frac{10}{x} + 10$ **6**

[A separation of the form $\frac{a\,dx}{x^2(10-x)} = b\,dt$ is essential for the M1. The f.t. is on A, B, C]

[If A or B (D or E) omitted from the form of fractions, give B0M1A0A0 in (i); M1A1√ A1√M1A0 in (ii)]

44. O/N 08/P03/Q8

(i) State or obtain $\frac{dV}{dt} = 4h^2\frac{dh}{dt}$, or $\frac{dV}{dh} = 4h^2$, or equivalent

State or imply $\frac{dV}{dt} = 20 - kh^2$

Use the given values to evaluate k

Show that $k = 0.2$, or equivalent, and obtain the given equation

[The M1 is dependent on at least one B mark having been earned.] **[4]**

(ii) Fully justify the given identity **[1]**

(iii) Separate variables correctly and attempt integration of both sides

Obtain terms $-20h$ and t, or equivalent

Obtain terms $a\ln(10+h) + b\ln(10-h)$, where $ab \neq 0$, or $k\ln\left(\frac{10+h}{10-h}\right)$

Obtain correct terms, i.e. with $a = 100$ and $b = -100$, or $k = 2000/20$, or equivalent

Evaluate a constant and obtain a correct expression for t in terms of h **[5]**

45. M/J 08/P03/Q8

(i) State $\dfrac{y}{TN} = \dfrac{dy}{dx}$, or equivalent

Express area of PTN in terms of y and $\dfrac{dy}{dx}$, and equate to $\tan x$

Obtain given relation correctly [3]

(ii) Separate variables correctly

Integrate and obtain term $-\dfrac{2}{y}$, or equivalent

Integrate and obtain term $\ln(\sin x)$, or equivalent

Evaluate a constant or use limits $y = 2,\ x = \frac{1}{6}\pi$ in a solution containing a term of the form a/y or $b\ln(\sin x)$

Obtain correct solution in any form, e.g. $-\dfrac{2}{y} = \ln(2\sin x) - 1$

Rearrange as $y = 2/(1 - \ln(2\sin x))$, or equivalent [6]

[Allow decimals, e.g. as in a solution $y = 2/(0.3 - \ln(\sin x))$.]

46. O/N 07/P03/Q7

(i) Separate variables correctly and attempt integration of both sides

Obtain term $\ln N$, or equivalent

Obtain term $\dfrac{k}{0.02}\sin(0.02t)$, or equivalent

Use $t = 0$, $N = 125$ to evaluate a constant, or as limits, in a solution containing terms of the form $a\ln N$ and $b\sin(0.02t)$, or equivalent [5]

Obtain any correct form of solution, e.g. $\ln N = 50k\sin(0.02t) + \ln 125$

(ii) Substituting $N = 166$ and $t = 30$, evaluate k [2]

Obtain $k = 0.0100479\ldots$(accept $k = 0.01$)

(iii) Rearrange and obtain $N = 125\exp(0.502\sin(0.02t))$, or equivalent

Set $\sin(0.02t) = -1$ in the expression for N, or equivalent [3]

Obtain least value 75.6 (accept answers in the interval [75, 76])

[For the B1, accept 0.5 following $k = 0.01$, and allow 4.8 or better for ln 125.]

47. M/J 07/P03/Q10

(i) State $\dfrac{dh}{dt} = k(9-h)^{\frac{1}{3}}$ [2]

Show that $k = 0.1$

(ii) Separate variables correctly and attempt integration of at least one side [7]

Obtain terms $-\dfrac{3}{2}(9-h)^{\frac{2}{3}}$ and 0.1 t, or equivalent

Evaluate a constant, or use limits t = 0, h = 1 with a solution containing terms of the form $a(9-h)^p$ and bt, where $p > 0$

Obtain solution in any form e.g. $-\dfrac{3}{2}(9-h)^{\frac{2}{3}} = 0.1t - 6$

Rearrange and make h the subject

Obtain answer $h = 9 - \left(4 - \dfrac{1}{15}t\right)^{\frac{3}{2}}$, or equivalent

(iii) State that the maximum height is $h - 9$ [2]

State that the time taken is 60 years

(iv) Substitute $h - 9/2$ and obtain $t = 19.1$ (accept 19, 19.0 and 19.2) [1]

48. O/N 06/P03/Q4

Separate variables correctly and attempt to integrate one side

Obtain terms $\frac{1}{2}\ln\left(1+y^2\right)$ and x, or equivalent

Evaluate a constant or use limits x = 0, y = 2 with a solution containing terms $k\ln\left(1+y^2\right)$ and x. or equivalent

Obtain any correct form of solution e.g. $\frac{1}{2}\ln\left(1+y^2\right) = x + \frac{1}{2}\ln 5$

Rearrange and obtain $y^2 = 5e^{2x} - 1$, or equivalent **[6]**

49. M/J 06/P03/Q5

(i) State or imply that $\frac{dx}{dt} = kx - 25$

Show that $k = 0.1$ and justify the given statement **[2]**

(ii) Separate variables and attempt integration

Obtain $\ln(x - 250)$, or equivalent

Obtain $0.1t$, or equivalent

Evaluate a constant or use limits $t = 0$, x = 1000 with a solution containing terms $a\ln(x - 250)$ and bt

Obtain any correct form of solution, e.g. $\ln(x - 250) = 0.1t + \ln 750$

Rearrange and obtain $x = 250(3e^{0.1t} + 1)$, or equivalent **[6]**

50. O/N 05/P03/Q8

(i) Separate variables correctly and attempt to integrate both sides

Obtain term $\ln x$, or equivalent

Obtain term $-\frac{1}{2}kt^2$, or equivalent

Use $t = 0$, $x = 100$ to evaluate a constant, or as limits

Obtain solution in any correct form, e.g. $\ln x = -\frac{1}{2}kt^2 + \ln 100$ **[5]**

(ii) Use $t = 20$, $x = 90$ to obtain an equation in k

Substitute $x = 50$ and attempt to obtain an unsimplified numerical expression for t^2, such as $t^2 = 400(\ln 100 - \ln 50)/(\ln 100 - \ln 90)$

Obtain answer $t = 51.3$ **[3]**

51. M/J 05/P03/Q8

(i) Attempt to express integrand in partial fractions,

e.g. obtain A or B in $\frac{A}{y} + \frac{B}{4-y}$

Obtain $\frac{1}{4}\left(\frac{1}{y} + \frac{1}{4-y}\right)$, or equivalent

Integrate and obtain $\frac{1}{4}\ln y - \frac{1}{4}\ln(4-y)$, or equivalent [4]

(ii) Separate variables correctly, integrate $\frac{A}{y} + \frac{B}{4-y}$ and obtain further term x, or equivalent

Use $y = 1$ and $x = 0$ to evaluate a constant, or as limits

Obtain answer in any correct form

Obtain final answer $y = 4/(3e^{-4x} + 1)$, or equivalent [4]

(iii) State that y approaches 4 as x becomes very large [1]

52. O/N 04/P03/Q10

(i) State or imply $\frac{dV}{dt} = 1000\frac{dh}{dt}$

State or imply $\frac{dV}{dt} = 30 - k\sqrt{h}$ or $\frac{dh}{dt} = 0.03 - m\sqrt{h}$ 3

Show that $k = 10$ or $m = 0.01$ and justify the given equation

[Allow the first B1 for the statement that 0.03 = 30/1000.]

(ii) Separate variables and attempt integration of $\frac{x-3}{x}$ with respect to x

Obtain $x - 3\ln x$, or equivalent

Obtain $0.005t$, or equivalent

Use $x = 3$, $t = 0$ in the evaluation of a constant or as limits in an answer involving $\ln x$ and kt

Obtain answer in any correct form e.g. $t = 200(x - 3 - 3\ln x + 3\ln 3)$

[To qualify for the first M mark, an attempt to solve the earlier differential equation in h and t must involve correct separation of variables, the use of a substitution such as $\sqrt{h} = u$, and an attempt to integrate the resulting function of u.] 5

(iii) Substitute $x = 1$ and calculate t 2

Obtain answer $t = 259$ correctly

53. M/J 04/P03/Q6

Separate variables and attempt to integrate

Obtain terms $\frac{1}{3}\ln(y^3+1)$ and x, or equivalent

Evaluate a constant or use limits $x = 0$, $y = 1$ with a solution containing terms $k\ln(y^3+1)$ and x, or equivalent

Obtain any correct form of solution e.g. $\frac{1}{3}\ln(y^3+1) = x + \frac{1}{3}\ln 2$ 6

Rearrange and obtain $y = (2e^{3x} - 1)^{\frac{1}{3}}$, or equivalent

[f.t. is on $k \neq 0$.]

54. O/N 03/P03/Q9

(i) Separate variables and attempt to integrate $\frac{1}{\sqrt{(P-A)}}$

Obtain term $2\sqrt{(P-A)}$ [3]

Obtain term $-kt$

(ii) Use limits $P = 5A$, $t = 0$ and attempt to find constant c

Obtain $c = 4\sqrt{A}$, or equivalent

Use limits $P = 2A$, $t = 2$ and attempt to find k [4]

Obtain given answer $k = \sqrt{A}$ correctly

(iii) Substitute $P = A$ and attempt to calculate t

Obtain answer $t = 4$ [2]

(iv) Using answers to part **(ii)**, attempt to rearrange solution to give P in terms of A and t

Obtain $P = \frac{1}{4}A(4 + (4-t)^2)$, or equivalent, having squared $\sqrt{A}$ [2]

[For the M1, $\sqrt{(P-A)}$ must be treated correctly.]

55. M/J 03/P03/Q7

(i) State or imply that $\frac{dx}{dt} = k(100 - x)$

Justify $k = 0.02$ **[2]**

(ii) Separate variables and attempt to integrate $\frac{1}{100-x}$

Obtain term $-\ln(100 - x)$, or equivalent

Obtain term $0.02t$, or equivalent

Use $x = 5$, $t = 0$ to evaluate a constant, or as limits

Obtain correct answer in any form, e.g. $-\ln(100 - x) = 0.02t - \ln 95$

Rearrange to give x in terms of t in any correct form,

e.g. $x = 100 - 95\exp(-0.02t)$

[SR: $\ln(100 - x)$ for $-\ln(100 - x)$. If no other error and $x = 100 - 95\exp(0.02t)$ or equivalent obtained, give M1A0A1M1A0A1√] **[6]**

(iii) State that x tends to 100 as t becomes very large **[1]**

56. O/N 02/P03/Q9

(i) State or imply that $\frac{da}{dt} = ka(10 - a)$

Justify $k = 0.004$ **[2]**

(ii) Resolve $\frac{1}{a(10-a)}$ into partial fractions $\frac{A}{a} + \frac{B}{10-a}$ and obtain values $A = B = \frac{1}{10}$

Separate variables obtaining $\int \frac{da}{a(10-a)} = \int k\,dt$ and attempt to integrate both sides

Obtain $\frac{1}{10}\ln a - \frac{1}{10}\ln(10 - a)$

Obtain $0.004t$, or equivalent

Evaluate a constant, or use limits $t = 0$, $a = 5$

Obtain answer $t = 25\ln\left(\frac{a}{10-a}\right)$, or equivalent **[6]**

(iii) Substitute $a = 9$ and calculate t

Obtain answer $t = 54.9$ or 55

[Substitution of $a = 0.9$ scores M0] **[2]**

57. M/J 02/P03/Q7

(i) State that $\frac{dm}{dt} = k(50 - m)^2$

Justify $k = 0.002$ **[2]**

(ii) Separate variables and attempt to integrate $\frac{1}{(50-m)^2}$

Obtain $\pm\frac{1}{(50-m)}$ and $0.002t$, or equivalent

Evaluate a constant or use limits $t = 0$, $m = 0$

Obtain any correct form of solution e.g. $\frac{1}{(50-m)} = 0.002t + \frac{1}{50}$

Obtain given answer correctly **[5]**

(iii) Obtain answer $m = 25$ when $t = 10$

Obtain answer $t = 90$ when $m = 45$ **[2]**

(iv) State that m approaches 50 **[1]**

Unit-9: Complex Numbers

1. M/J 18/P32/Q7

The complex numbers $-3\sqrt{3} + i$ and $\sqrt{3} + 2i$ are denoted by u and v respectively.

(i) Find, in the form $x + iy$, where x and y are real and exact, the complex numbers uv and $\frac{u}{v}$. [5]

(ii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers u and v respectively. Prove that angle $AOB = \frac{2}{3}\pi$. [3]

2. M/J 18/P31/Q7

(i) Showing all working and without using a calculator, solve the equation $z^2 + (2\sqrt{6})z + 8 = 0$, giving your answers in the form $x + iy$, where x and y are real and exact. [3]

(ii) Sketch an Argand diagram showing the points representing the roots. [1]

(iii) The points representing the roots are A and B, and O is the origin. Find angle AOB. [3]

(iv) Prove that triangle AOB is equilateral. [1]

3. M/J 18/P33/Q9

(a) Find the complex number z satisfying the equation

$$3z - iz^* = 1 + 5i,$$

where z^* denotes the complex conjugate of z. [4]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z| \leqslant 3$ and $\text{Im}\, z \geqslant 2$, where $\text{Im}\, z$ denotes the imaginary part of z. Calculate the greatest value of $\arg z$ for points in this region. Give your answer in radians correct to 2 decimal places. [5]

4. O/N 17/P32/Q7

The complex number $1 - (\sqrt{3})i$ is denoted by u.

(i) Find the modulus and argument of u. [2]

(ii) Show that $u^3 + 8 = 0$. [2]

(iii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z - u| \leqslant 2$ and $\text{Re}\, z \geqslant 2$, where $\text{Re}\, z$ denotes the real part of z. [4]

5. O/N 17/P31/Q7, O/N 17/P33/Q7

(a) The complex number u is given by $u = 8 - 15i$. Showing all necessary working, find the two square roots of u. Give answers in the form $a + ib$, where the numbers a and b are real and exact. [5]

(b) On an Argand diagram, shade the region whose points represent complex numbers satisfying both the inequalities $|z - 2 - i| \leqslant 2$ and $0 \leqslant \arg(z - i) \leqslant \frac{1}{4}\pi$. [4]

6. M/J 17/P32/Q6

The complex number $2 - i$ is denoted by u.

(i) It is given that u is a root of the equation $x^3 + ax^2 - 3x + b = 0$, where the constants a and b are real. Find the values of a and b. [4]

(ii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z - u| < 1$ and $|z| < |z + i|$. [4]

7. M/J 17/P31/Q7

The complex numbers u and w are defined by $u = -1 + 7i$ and $w = 3 + 4i$.

(i) Showing all your working, find in the form $x + iy$, where x and y are real, the complex numbers $u - 2w$ and $\frac{u}{w}$. [4]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, w and $u - 2w$ respectively.

(ii) Prove that angle $AOB = \frac{1}{4}\pi$. [2]

(iii) State fully the geometrical relation between the line segments OB and CA. [2]

8. M/J 17/P33/Q11

(a) The complex numbers z and w satisfy the equations

$$z + (1 + i)w = i \qquad \text{and} \qquad (1 - i)z + iw = 1.$$

Solve the equations for z and w, giving your answers in the form $x + iy$, where x and y are real. [6]

(b) The complex numbers u and v are given by $u = 1 + (2\sqrt{3})i$ and $v = 3 + 2i$. In an Argand diagram, u and v are represented by the points A and B. A third point C lies in the first quadrant and is such that $BC = 2AB$ and angle $ABC = 90°$. Find the complex number z represented by C, giving your answer in the form $x + iy$, where x and y are real and exact. [4]

9. O/N 16/P32/Q9, O/N 16/P31/Q9

(a) Solve the equation $(1 + 2i)w^2 + 4w - (1 - 2i) = 0$, giving your answers in the form $x + iy$, where x and y are real. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 1 - i| \leqslant 2$ and $-\frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{4}\pi$. [5]

10. O/N 16/P33/Q7

The complex number z is defined by $z = (\sqrt{2}) - (\sqrt{6})i$. The complex conjugate of z is denoted by z^*.

(i) Find the modulus and argument of z. [2]

(ii) Express each of the following in the form $x + iy$, where x and y are real and exact:

(a) $z + 2z^*$;

(b) $\dfrac{z^*}{iz}$. [4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z^* and iz respectively. Prove that angle AOB is equal to $\frac{1}{6}\pi$. [3]

11. M/J 16/P32/Q10

(a) Showing all necessary working, solve the equation $iz^2 + 2z - 3i = 0$, giving your answers in the form $x + iy$, where x and y are real and exact. [5]

(b) (i) On a sketch of an Argand diagram, show the locus representing complex numbers satisfying the equation $|z| = |z - 4 - 3i|$. [2]

(ii) Find the complex number represented by the point on the locus where $|z|$ is least. Find the modulus and argument of this complex number, giving the argument correct to 2 decimal places. [3]

12. M/J 16/P31/Q10

(a) Showing all your working and without the use of a calculator, find the square roots of the complex number $7 - (6\sqrt{2})i$. Give your answers in the form $x + iy$, where x and y are real and exact. [5]

(b) (i) On an Argand diagram, sketch the loci of points representing complex numbers w and z such that $|w - 1 - 2i| = 1$ and $\arg(z - 1) = \frac{3}{4}\pi$. [4]

(ii) Calculate the least value of $|w - z|$ for points on these loci. [2]

13. M/J 16/P33/Q9

The complex numbers $-1 + 3i$ and $2 - i$ are denoted by u and v respectively. In an Argand diagram with origin O, the points A, B and C represent the numbers u, v and $u + v$ respectively.

(i) Sketch this diagram and state fully the geometrical relationship between OB and AC. [4]

(ii) Find, in the form $x + iy$, where x and y are real, the complex number $\dfrac{u}{v}$. [3]

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

14. O/N 15/P32/Q9, O/N 15/P31/Q9

The complex number $3 - \mathrm{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

(i) On an Argand diagram with origin O, show the points A, B and C representing the complex numbers u, u^* and $u^* - u$ respectively. What type of quadrilateral is $OABC$? [4]

(ii) Showing your working and without using a calculator, express $\frac{u^*}{u}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(iii) By considering the argument of $\frac{u^*}{u}$, prove that

$$\tan^{-1}\left(\tfrac{3}{4}\right) = 2\tan^{-1}\left(\tfrac{1}{3}\right).$$ [3]

15. O/N 15/P33/Q9

(a) It is given that $(1 + 3\mathrm{i})w = 2 + 4\mathrm{i}$. Showing all necessary working, prove that the exact value of $|w^2|$ is 2 and find $\arg(w^2)$ correct to 3 significant figures. [6]

(b) On a single Argand diagram sketch the loci $|z| = 5$ and $|z - 5| = |z|$. Hence determine the complex numbers represented by points common to both loci, giving each answer in the form $re^{i\theta}$. [4]

16. M/J 15/P32/Q7

The complex number u is given by $u = -1 + (4\sqrt{3})\mathrm{i}$.

(i) Without using a calculator and showing all your working, find the two square roots of u. Give your answers in the form $a + \mathrm{i}b$, where the real numbers a and b are exact. [5]

(ii) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying the relation $|z - u| = 1$. Determine the greatest value of $\arg z$ for points on this locus. [4]

17. M/J 15/P31/Q8

The complex number w is defined by $w = \frac{22 + 4\mathrm{i}}{(2 - \mathrm{i})^2}$.

(i) Without using a calculator, show that $w = 2 + 4\mathrm{i}$. [3]

(ii) It is given that p is a real number such that $\frac{1}{4}\pi \leqslant \arg(w + p) \leqslant \frac{3}{4}\pi$. Find the set of possible values of p. [3]

(iii) The complex conjugate of w is denoted by w^*. The complex numbers w and w^* are represented in an Argand diagram by the points S and T respectively. Find, in the form $|z - a| = k$, the equation of the circle passing through S, T and the origin. [3]

18. M/J 15/P33/Q8

The complex number $1 - \mathrm{i}$ is denoted by u.

(i) Showing your working and without using a calculator, express

$$\frac{\mathrm{i}}{u}$$

in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) On an Argand diagram, sketch the loci representing complex numbers z satisfying the equations $|z - u| = |z|$ and $|z - \mathrm{i}| = 2$. [4]

(iii) Find the argument of each of the complex numbers represented by the points of intersection of the two loci in part **(ii)**. [3]

19. O/N 14/P32/Q5, O/N 14/P31/Q5

The complex numbers w and z satisfy the relation

$$w = \frac{z + \mathrm{i}}{\mathrm{i}z + 2}.$$

(i) Given that $z = 1 + \mathrm{i}$, find w, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [4]

(ii) Given instead that $w = z$ and the real part of z is negative, find z, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [4]

20. O/N 14/P33/Q5

The complex numbers w and z are defined by $w = 5 + 3i$ and $z = 4 + i$.

(i) Express $\frac{iw}{z}$ in the form $x + iy$, showing all your working and giving the exact values of x and y. [3]

(ii) Find wz and hence, by considering arguments, show that

$$\tan^{-1}\left(\tfrac{3}{5}\right) + \tan^{-1}\left(\tfrac{1}{4}\right) = \tfrac{1}{4}\pi.$$ [4]

21. M/J 14/P32/Q7

(a) It is given that $-1 + (\sqrt{5})i$ is a root of the equation $z^3 + 2z + a = 0$, where a is real. Showing your working, find the value of a, and write down the other complex root of this equation. [4]

(b) The complex number w has modulus 1 and argument 2θ radians. Show that $\frac{w-1}{w+1} = i\tan\theta$. [4]

22. M/J 14/P31/Q5

The complex number z is defined by $z = \frac{9\sqrt{3} + 9i}{\sqrt{3} - i}$. Find, showing all your working,

(i) an expression for z in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, [5]

(ii) the two square roots of z, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

23. M/J 14/P33/Q7

(a) The complex number $\frac{3-5i}{1+4i}$ is denoted by u. Showing your working, express u in the form $x + iy$, where x and y are real. [3]

(b) (i) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 2 - i| \leqslant 1$ and $|z - i| \leqslant |z - 2|$. [4]

(ii) Calculate the maximum value of $\arg z$ for points lying in the shaded region. [2]

24. O/N 13/P32/Q8

(a) The complex numbers u and v satisfy the equations

$$u + 2v = 2i \quad \text{and} \quad iu + v = 3.$$

Solve the equations for u and v, giving both answers in the form $x + iy$, where x and y are real. [5]

(b) On an Argand diagram, sketch the locus representing complex numbers z satisfying $|z + i| = 1$ and the locus representing complex numbers w satisfying $\arg(w - 2) = \frac{3}{4}\pi$. Find the least value of $|z - w|$ for points on these loci. [5]

25. O/N 13/P33/Q9

(a) Without using a calculator, use the formula for the solution of a quadratic equation to solve

$$(2 - i)z^2 + 2z + 2 + i = 0.$$

Give your answers in the form $a + bi$. [5]

(b) The complex number w is defined by $w = 2e^{\frac{1}{4}\pi i}$. In an Argand diagram, the points A, B and C represent the complex numbers w, w^3 and w^* respectively (where w^* denotes the complex conjugate of w). Draw the Argand diagram showing the points A, B and C, and calculate the area of triangle ABC. [5]

26. M/J 13/P32/Q9

(a) The complex number w is such that $\text{Re}\, w > 0$ and $w + 3w^* = iw^2$, where w^* denotes the complex conjugate of w. Find w, giving your answer in the form $x + iy$, where x and y are real. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z - 2i| \leqslant 2$ and $0 \leqslant \arg(z + 2) \leqslant \frac{1}{4}\pi$. Calculate the greatest value of $|z|$ for points in this region, giving your answer correct to 2 decimal places. [6]

27. M/J 13/P31/Q7

(a) Without using a calculator, solve the equation

$$3w + 2iw^* = 17 + 8i,$$

where w^* denotes the complex conjugate of w. Give your answer in the form $a + bi$. [4]

(b) In an Argand diagram, the loci

$$\arg(z - 2i) = \tfrac{1}{6}\pi \quad \text{and} \quad |z - 3| = |z - 3i|$$

intersect at the point P. Express the complex number represented by P in the form $re^{i\theta}$, giving the exact value of θ and the value of r correct to 3 significant figures. [5]

28. M/J 13/P33/Q7

The complex number z is defined by $z = a + ib$, where a and b are real. The complex conjugate of z is denoted by z^*.

(i) Show that $|z|^2 = zz^*$ and that $(z - ki)^* = z^* + ki$, where k is real. [2]

In an Argand diagram a set of points representing complex numbers z is defined by the equation $|z - 10i| = 2|z - 4i|$.

(ii) Show, by squaring both sides, that

$$zz^* - 2iz^* + 2iz - 12 = 0.$$

Hence show that $|z - 2i| = 4$. [5]

(iii) Describe the set of points geometrically. [1]

29. O/N 12/P33/Q10

(a) Without using a calculator, solve the equation $iw^2 = (2 - 2i)^2$. [3]

(b) (i) Sketch an Argand diagram showing the region R consisting of points representing the complex numbers z where

$$|z - 4 - 4i| \leqslant 2.$$ [2]

(ii) For the complex numbers represented by points in the region R, it is given that

$$p \leqslant |z| \leqslant q \quad \text{and} \quad \alpha \leqslant \arg z \leqslant \beta.$$

Find the values of p, q, α and β, giving your answers correct to 3 significant figures. [6]

30. M/J 12/P32/Q7

The complex number u is defined by

$$u = \frac{1 + 2i}{1 - 3i}.$$

(i) Express u in the form $x + iy$, where x and y are real. [3]

(ii) Show on a sketch of an Argand diagram the points A, B and C representing the complex numbers u, $1 + 2i$ and $1 - 3i$ respectively. [2]

(iii) By considering the arguments of $1 + 2i$ and $1 - 3i$, show that

$$\tan^{-1}2 + \tan^{-1}3 = \tfrac{3}{4}\pi.$$ [3]

31. M/J 12/P31/Q4

The complex number u is defined by $u = \dfrac{(1 + 2i)^2}{2 + i}$.

(i) Without using a calculator and showing your working, express u in the form $x + iy$, where x and y are real. [4]

(ii) Sketch an Argand diagram showing the locus of the complex number z such that $|z - u| = |u|$. [3]

32. M/J 12/P33/Q10

(a) The complex numbers u and w satisfy the equations

$$u - w = 4i \quad \text{and} \quad uw = 5.$$

Solve the equations for u and w, giving all answers in the form $x + iy$, where x and y are real. [5]

(b) (i) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z-2+2i| \leqslant 2$, $\arg z \leqslant -\frac{1}{4}\pi$ and $\operatorname{Re} z \geqslant 1$, where $\operatorname{Re} z$ denotes the real part of z. [5]

(ii) Calculate the greatest possible value of $\operatorname{Re} z$ for points lying in the shaded region. [1]

33. O/N 11/P32/Q10, O/N 11/P31/Q10

(a) Showing your working, find the two square roots of the complex number $1-(2\sqrt{6})i$. Give your answers in the form $x+iy$, where x and y are exact. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent the complex numbers z which satisfy the inequality $|z-3i| \leqslant 2$. Find the greatest value of $\arg z$ for points in this region. [5]

34. O/N 11/P33/Q6

The complex number w is defined by $w=-1+i$.

(i) Find the modulus and argument of w^2 and w^3, showing your working. [4]

(ii) The points in an Argand diagram representing w and w^2 are the ends of a diameter of a circle. Find the equation of the circle, giving your answer in the form $|z-(a+bi)|=k$. [4]

35. M/J 11/P32/Q7

(a) The complex number u is defined by $u=\dfrac{5}{a+2i}$, where the constant a is real.

(i) Express u in the form $x+iy$, where x and y are real. [2]

(ii) Find the value of a for which $\arg(u^*)=\frac{3}{4}\pi$, where u^* denotes the complex conjugate of u. [3]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z|<2$ and $|z|<|z-2-2i|$. [4]

36. M/J 11/P31/Q8

The complex number u is defined by $u=\dfrac{6-3i}{1+2i}$.

(i) Showing all your working, find the modulus of u and show that the argument of u is $-\frac{1}{2}\pi$. [4]

(ii) For complex numbers z satisfying $\arg(z-u)=\frac{1}{4}\pi$, find the least possible value of $|z|$. [3]

(iii) For complex numbers z satisfying $|z-(1+i)u|=1$, find the greatest possible value of $|z|$. [3]

37. M/J 11/P33/Q7

(i) Find the roots of the equation

$$z^2+(2\sqrt{3})z+4=0,$$

giving your answers in the form $x+iy$, where x and y are real. [2]

(ii) State the modulus and argument of each root. [3]

(iii) Showing all your working, verify that each root also satisfies the equation

$$z^6=-64.$$ [3]

38. O/N 10/P32/Q6

The complex number z is given by

$$z=(\sqrt{3})+i.$$

(i) Find the modulus and argument of z. [2]

(ii) The complex conjugate of z is denoted by z^*. Showing your working, express in the form $x+iy$, where x and y are real,

(a) $2z+z^*$,

(b) $\dfrac{iz^*}{z}$. [4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z and iz^* respectively. Prove that angle $AOB=\frac{1}{6}\pi$. [3]

39. O/N 10/P33/Q3

The complex number w is defined by $w = 2 + \mathrm{i}$.

(i) Showing your working, express w^2 in the form $x + \mathrm{i}y$, where x and y are real. Find the modulus of w^2. [3]

(ii) Shade on an Argand diagram the region whose points represent the complex numbers z which satisfy

$$|z - w^2| \leqslant |w^2|.$$ [3]

40. M/J 10/P32/Q8

The variable complex number z is given by

$$z = 1 + \cos 2\theta + \mathrm{i}\sin 2\theta,$$

where θ takes all values in the interval $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(i) Show that the modulus of z is $2\cos\theta$ and the argument of z is θ. [6]

(ii) Prove that the real part of $\frac{1}{z}$ is constant. [3]

41. M/J 10/P31/Q7

The complex number $2 + 2\mathrm{i}$ is denoted by u.

(i) Find the modulus and argument of u. [2]

(ii) Sketch an Argand diagram showing the points representing the complex numbers 1, i and u. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1| \leqslant |z - \mathrm{i}|$ and $|z - u| \leqslant 1$. [4]

(iii) Using your diagram, calculate the value of $|z|$ for the point in this region for which $\arg z$ is least. [3]

42. M/J 10/P33/Q8

(a) The equation $2x^3 - x^2 + 2x + 12 = 0$ has one real root and two complex roots. Showing your working, verify that $1 + \mathrm{i}\sqrt{3}$ is one of the complex roots. State the other complex root. [4]

(b) On a sketch of an Argand diagram, show the point representing the complex number $1 + \mathrm{i}\sqrt{3}$. On the same diagram, shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1 - \mathrm{i}\sqrt{3}| \leqslant 1$ and $\arg z \leqslant \frac{1}{3}\pi$. [5]

43. O/N 09/P32/Q7

The complex numbers $-2 + \mathrm{i}$ and $3 + \mathrm{i}$ are denoted by u and v respectively.

(i) Find, in the form $x + \mathrm{i}y$, the complex numbers

(a) $u + v$, [1]

(b) $\frac{u}{v}$, showing all your working. [3]

(ii) State the argument of $\frac{u}{v}$. [1]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, v and $u + v$ respectively.

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

(iv) State fully the geometrical relationship between the line segments OA and BC. [2]

44. O/N 09/P31/Q7

The complex number $-2 + \mathrm{i}$ is denoted by u.

(i) Given that u is a root of the equation $x^3 - 11x - k = 0$, where k is real, find the value of k. [3]

(ii) Write down the other complex root of this equation. [1]

(iii) Find the modulus and argument of u. [2]

(iv) Sketch an Argand diagram showing the point representing u. Shade the region whose points represent the complex numbers z satisfying both the inequalities

$$|z| < |z - 2| \quad \text{and} \quad 0 < \arg(z - u) < \tfrac{1}{4}\pi.$$ [4]

45. M/J 09/P03/Q7

(i) Solve the equation $z^2 + (2\sqrt{3})iz - 4 = 0$, giving your answers in the form $x + iy$, where x and y are real. [3]

(ii) Sketch an Argand diagram showing the points representing the roots. [1]

(iii) Find the modulus and argument of each root. [3]

(iv) Show that the origin and the points representing the roots are the vertices of an equilateral triangle. [1]

46. O/N 08/P03/Q10

The complex number w is given by $w = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

(i) Find the modulus and argument of w. [2]

(ii) The complex number z has modulus R and argument θ, where $-\frac{1}{3}\pi < \theta < \frac{1}{3}\pi$. State the modulus and argument of wz and the modulus and argument of $\frac{z}{w}$. [4]

(iii) Hence explain why, in an Argand diagram, the points representing z, wz and $\frac{z}{w}$ are the vertices of an equilateral triangle. [2]

(iv) In an Argand diagram, the vertices of an equilateral triangle lie on a circle with centre at the origin. One of the vertices represents the complex number $4 + 2i$. Find the complex numbers represented by the other two vertices. Give your answers in the form $x + iy$, where x and y are real and exact. [4]

47. M/J 08/P03/Q5

The variable complex number z is given by

$$z = 2\cos\theta + i(1 - 2\sin\theta),$$

where θ takes all values in the interval $-\pi < \theta \leqslant \pi$.

(i) Show that $|z - i| = 2$, for all values of θ. Hence sketch, in an Argand diagram, the locus of the point representing z. [3]

(ii) Prove that the real part of $\frac{1}{z + 2 - i}$ is constant for $-\pi < \theta < \pi$. [4]

48. O/N 07/P03/Q8

(a) The complex number z is given by $z = \frac{4 - 3i}{1 - 2i}$.

(i) Express z in the form $x + iy$, where x and y are real. [2]

(ii) Find the modulus and argument of z. [2]

(b) Find the two square roots of the complex number $5 - 12i$, giving your answers in the form $x + iy$, where x and y are real. [6]

49. M/J 07/P03/Q8

The complex number $\frac{2}{-1 + i}$ is denoted by u.

(i) Find the modulus and argument of u and u^2. [6]

(ii) Sketch an Argand diagram showing the points representing the complex numbers u and u^2. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z| < 2$ and $|z - u^2| < |z - u|$. [4]

50. O/N 06/P03/Q9

The complex number u is given by

$$u = \frac{3 + i}{2 - i}.$$

(i) Express u in the form $x + iy$, where x and y are real. [3]

(ii) Find the modulus and argument of u. [2]

(iii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the point representing the complex number z such that $|z - u| = 1$. [3]

(iv) Using your diagram, calculate the least value of $|z|$ for points on this locus. [2]

51. M/J 06/P03/Q7

The complex number $2 + \mathrm{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

(i) Show, on a sketch of an Argand diagram with origin O, the points A, B and C representing the complex numbers u, u^* and $u + u^*$ respectively. Describe in geometrical terms the relationship between the four points O, A, B and C. [4]

(ii) Express $\dfrac{u}{u^*}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(iii) By considering the argument of $\dfrac{u}{u^*}$, or otherwise, prove that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = 2\tan^{-1}\left(\tfrac{1}{2}\right).$$ [2]

52. O/N 05/P03/Q7

The equation $2x^3 + x^2 + 25 = 0$ has one real root and two complex roots.

(i) Verify that $1 + 2\mathrm{i}$ is one of the complex roots. [3]

(ii) Write down the other complex root of the equation. [1]

(iii) Sketch an Argand diagram showing the point representing the complex number $1 + 2\mathrm{i}$. Show on the same diagram the set of points representing the complex numbers z which satisfy

$$|z| = |z - 1 - 2\mathrm{i}|.$$ [4]

53. M/J 05/P03/Q3

(i) Solve the equation $z^2 - 2\mathrm{i}z - 5 = 0$, giving your answers in the form $x + \mathrm{i}y$ where x and y are real. [3]

(ii) Find the modulus and argument of each root. [3]

(iii) Sketch an Argand diagram showing the points representing the roots. [1]

54. O/N 04/P03/Q6

The complex numbers $1 + 3\mathrm{i}$ and $4 + 2\mathrm{i}$ are denoted by u and v respectively.

(i) Find, in the form $x + \mathrm{i}y$, where x and y are real, the complex numbers $u - v$ and $\dfrac{u}{v}$. [3]

(ii) State the argument of $\dfrac{u}{v}$. [1]

In an Argand diagram, with origin O, the points A, B and C represent the numbers u, v and $u - v$ respectively.

(iii) State fully the geometrical relationship between OC and BA. [2]

(iv) Prove that angle $AOB = \frac{1}{4}\pi$ radians. [2]

55. M/J 04/P03/Q8

(i) Find the roots of the equation $z^2 - z + 1 = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Obtain the modulus and argument of each root. [3]

(iii) Show that each root also satisfies the equation $z^3 = -1$. [2]

56. O/N 03/P03/Q7

The complex number u is given by $u = \dfrac{7 + 4\mathrm{i}}{3 - 2\mathrm{i}}$.

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(ii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the complex number z such that $|z-u|=2$. [3]

(iii) Find the greatest value of arg z for points on this locus. [3]

57. M/J 03/P03/Q5

The complex number $2i$ is denoted by u. The complex number with modulus 1 and argument $\frac{2}{3}\pi$ is denoted by w.

(i) Find in the form $x+iy$, where x and y are real, the complex numbers w, uw and $\frac{u}{w}$. [4]

(ii) Sketch an Argand diagram showing the points U, A and B representing the complex numbers u, uw and $\frac{u}{w}$ respectively. [2]

(iii) Prove that triangle UAB is equilateral. [2]

58. O/N 02/P03/Q8

(a) Find the two square roots of the complex number $-3+4i$, giving your answers in the form $x+iy$, where x and y are real. [5]

(b) The complex number z is given by

$$z=\frac{-1+3i}{2+i}.$$

(i) Express z in the form $x+iy$, where x and y are real. [2]

(ii) Show on a sketch of an Argand diagram, with origin O, the points A, B and C representing the complex numbers $-1+3i$, $2+i$ and z respectively. [1]

(iii) State an equation relating the lengths OA, OB and OC. [1]

59. M/J 02/P03/Q9

The complex number $1+i\sqrt{3}$ is denoted by u.

(i) Express u in the form $r(\cos\theta+i\sin\theta)$, where $r>0$ and $-\pi<\theta\leq\pi$. Hence, or otherwise, find the modulus and argument of u^2 and u^3. [5]

(ii) Show that u is a root of the equation $z^2-2z+4=0$, and state the other root of this equation. [2]

(iii) Sketch an Argand diagram showing the points representing the complex numbers i and u. Shade the region whose points represent every complex number z satisfying both the inequalities

$$|z-i|\leq 1 \quad \text{and} \quad \arg z\geq \arg u.$$ [4]

Answers Section

1. M/J 18/P32/Q7

(i) Substitute in uv, expand the product and use $i^2=-1$

Obtain answer $uv = -11-5\sqrt{3}i$

EITHER: Substitute in u/v and multiply numerator and denominator by the conjugate of v, or equivalent

Obtain numerator $-7+7\sqrt{3}i$ or denominator 7

Obtain final answer $-1+\sqrt{3}i$

OR: Substitute in u/v, equate to $x+iy$ and solve for x or for y

Obtain $x=-1$ or $y=\sqrt{3}$

Obtain final answer $-1+\sqrt{3}\,i$ 5

(ii) Show the points A and B representing u and v in relatively correct positions

Carry out a complete method for finding angle AOB, e.g. calculate arg(u/v)

If using $\theta=\tan^{-1}\left(-\sqrt{3}\right)$ must refer to $\arg\left(\frac{u}{v}\right)$

Prove the given statement 3

2. M/J 18/P31/Q7

(i) Use quadratic formula, or completing the square, or the substitution $z=x+iy$ to find a root, using $i^2=-1$

Obtain a root, e.g. $-\sqrt{6}-\sqrt{2i}$

Obtain the other root, e.g. $-\sqrt{6}-\sqrt{2i}$ 3

(ii) Represent both roots in relatively correct positions 1

(iii) State or imply correct value of a relevant length or angle, e.g. OA, OB, AB, angle between OA or OB and the real axis

Carry out a complete method for finding angle OAB

Obtain $AOB=60°$ correctly 3

(iv) Give a complete justification of the given statement 1

3. M/J 18/P33/Q9

(a) Substitute and obtain a correct equation in x and y

Use $i^2=-1$ and equate real and imaginary parts

Obtain two correct equations in x and y, e.g.

$3x-y=1$ and $3y-x=5$

Solve and obtain answer $z=1+2(i)$ 4

(b) Show a circle with radius 3

Show the line $y=2$ extending in both quadrants

Shade the correct region

Carry out a complete method for finding the greatest value of arg z

Obtain answer 2.41 5

4. O/N 17/P32/Q7

(i) State modulus 2

State argument $-\frac{1}{3}\pi$ or $-60°$ ($\frac{5}{3}\pi$ or $300°$) 2

(ii) *EITHER:* Expand $(1-(\sqrt{3})i)^3$ completely and process i^2 and i^3

Verify that the given relation is satisfied

OR: $u^3 = 2^3(\cos(-\pi)+i\sin(-\pi))$ or equivalent: follow their answers to (i)

Verify that the given relation is satisfied 2

(iii) Show a circle with centre $1-(\sqrt{3})i$ in a relatively correct position

Show a circle with radius 2 passing through the origin

Show the line Re $z = 2$

Shade the correct region 4

5. O/N 17/P31/Q7, O/N 17/P33/Q7

(a) Square $x + iy$ and equate real and imaginary parts to 8 and -15

Obtain $x^2 - y^2 = 8$ and $2xy = -15$

Eliminate one unknown and find a horizontal equation in the other

Obtain $4x^4 - 32x^2 - 225 = 0$ or $4y^4 + 32y^2 - 225 = 0$, or three term equivalent

Obtain answers $\pm\frac{1}{\sqrt{2}}(5-3i)$ or equivalent 5

(b) Show a circle with centre $2+i$ in a relatively correct position

Show a circle with radius 2 and centre not at the origin

Show line through i at an angle of $\frac{1}{4}\pi$ to the real axis

Shade the correct region 4

6. M/J 17/P32/Q6

(i) *EITHER:*

Substitute $x = 2-i$ (or $x = 2+i$) in the equation and attempt expansions of x^2 and x^3

Equate real and/or imaginary parts to zero

Obtain $a = -2$

Obtain $b = 10$

OR1:

Substitute $x = 2-i$ in the equation and attempt expansions of x^2 and x^3

Substitute $x = 2+i$ in the equation and add/subtract the two equations

Obtain $a = -2$

Obtain $b = 10$

OR2:

Factorise to obtain $(x-2+i)(x-2-i)(x-p)$ $\left(=(x^2-4x+5)(x-p)\right)$

Compare coefficients

Obtain $a = -2$

Obtain $b = 10$

OR3:

Obtain the quadratic factor (x^2-4x+5)

Use algebraic division to obtain a real linear factor of the form $x-p$ and set the remainder equal to zero

Obtain $a = -2$

Obtain $b = 10$

OR4:
Use $\alpha\beta = 5$ and $\alpha + \beta = 4$ in $\alpha\beta + \beta\gamma + \gamma\alpha = -3$
Solve for γ and use in $\alpha\beta\gamma = -b$ and/or $\alpha + \beta + \gamma = -a$
Obtain $a = -2$
Obtain $b = 10$

OR5:
Factorise as $(x - (2-i))(x^2 + ex + g)$ and compare coefficients to form an equation in a and b
Equate real and/or imaginary parts to zero
Obtain $a = -2$ 4
Obtain $b = 10$

(ii) Show a circle with centre $2 - i$ in a relatively correct position
Show a circle with radius 1 and centre not at the origin
Show the perpendicular bisector of the line segment joining 0 to $-i$ 4
Shade the correct region

7. M/J 17/P31/Q7

(i) State that $u - 2w = -7 - i$
EITHER:
Multiply numerator and denominator of $\frac{u}{w}$ by $3 - 4i$, or equivalent
Simplify the numerator to $25 + 25i$ or denominator to 25
Obtain final answer $1 + i$
OR:
Obtain two equations in x and y and solve for x or for y
Obtain $x = 1$ or $y = 1$ 4
Obtain final answer $1 + i$

(ii) Find the argument of $\frac{u}{w}$ 2
Obtain the given answer

(iii) State that OB and CA are parallel 2
State that $CA = 2OB$, or equivalent

8. M/J 17/P33/Q11

(a) Solve for z or for w
Use $i^2 = -1$
Obtain $w = \frac{i}{2-i}$ or $z = \frac{2+i}{2-i}$
Multiply numerator and denominator by the conjugate of the denominator
Obtain $w = -\frac{1}{5} + \frac{2}{5}i$ 6
Obtain $z = \frac{3}{5} + \frac{4}{5}i$

(b) *EITHER:*
Find $\pm\left[2 + \left(2 - 2\sqrt{3}\right)i\right]$
Multiply by 2i (or $-2i$)
Add result to v
Obtain answer $4\sqrt{3} - 1 + 6i$
OR:
State $\frac{z - v}{v - u} = ki$, or equivalent
State $k = 2$
Substitute and solve for z even if i omitted 4
Obtain answer $4\sqrt{3} - 1 + 6i$

9. O/N 16/P32/Q9, O/N 16/P31/Q9

(a) EITHER: Use quadratic formula to solve for w

Use $i^2 = -1$

Obtain one of the answers $w = \frac{1}{2i+1}$ and $w = -\frac{5}{2i+1}$

Multiply numerator and denominator of an answer by $-2i + 1$, or equivalent

Obtain final answers $\frac{1}{5} - \frac{2}{5}i$ and $-1 + 2i$

OR1: Multiply the equation by $1 - 2i$

Use $i^2 = -1$

Obtain $5w^2 + 4w(1-2i) - (1-2i)^2 = 0$, or equivalent

Use quadratic formula or factorise to solve for w

Obtain final answers $\frac{1}{5} - \frac{2}{5}i$ and $-1 + 2i$

OR2: Substitute $w = x + iy$ and form equations for real and imaginary parts

Use $i^2 = -1$

Obtain $(x^2 - y^2) - 4xy + 4x - 1 = 0$ and $2(x^2 - y^2) + 2xy + 4y + 2 = 0$ o.e.

Form equation in x only or y only and solve [5]

Obtain final answers $\frac{1}{5} - \frac{2}{5}i$ and $-1 + 2i$

(b) Show a circle with centre $1 + i$

Show a circle with radius 2

Show half-line $\arg z = \frac{1}{4}\pi$

Show half-line $\arg z = -\frac{1}{4}\pi$ [5]

Shade the correct region

10. O/N 16/P33/Q7

(i) State modulus $2\sqrt{2}$, or equivalent [2]

State argument $-\frac{1}{3}\pi$ (or $-60°$)

(ii) (a) State answer $3\sqrt{2} + \sqrt{6}\,i$

(b) *EITHER:* Substitute for z and multiply numerator and denominator by conjugate of iz

Simplify the numerator to $4\sqrt{3} + 4i$ or the denominator to 8

Obtain final answer $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$

OR: Substitute for z, obtain two equations in x and y and solve for x or for y

Obtain $x = \frac{1}{2}\sqrt{3}$ or $y = \frac{1}{2}$, [4]

Obtain final answer $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$

(iii) Show points A and B in relatively correct positions

Carry out a complete method for finding angle AOB, e.g. calculate the argument of $\frac{z^*}{iz}$ [3]

Obtain the given answer

11. M/J 16/P32/Q10

(a) *EITHER*: Use quadratic formula to solve for z

Use $i^2 = -1$

Obtain a correct answer in any form, simplified as far as $(-2 \pm i\sqrt{8})/2i$

Multiply numerator and denominator by i, or equivalent

Obtain final answers $\sqrt{2} + i$ and $-\sqrt{2} + i$

OR: Substitute $x + iy$ and equate real and imaginary parts to zero
Use $i^2 = -1$
Obtain $-2xy + 2x = 0$ and $x^2 - y^2 + 2y - 3 = 0$, or equivalent [5]
Solve for x and y
Obtain final answers $\sqrt{2} + i$ and $-\sqrt{2} + i$

(b) (i) *EITHER*: Show the point representing $4 + 3i$ in relatively correct position [2]
Show the perpendicular bisector of the line segment joining this point to the origin
OR: Obtain correct Cartesian equation of the locus in any form, e.g. $8x + 6y = 25$
Show this line
[This f.t. is dependent on using a correct method to determine the equation.]

(ii) State or imply the relevant point is represented by $2 + 1.5i$ or is at $(2, 1.5)$ [3]
Obtain modulus 2.5
Obtain argument 0.64 (or 36.9°) (allow decimals in [0.64, 0.65] or [36.8, 36.9])

12. M/J 16/P31/Q10

(a) Square $x + iy$ and equate real and imaginary parts to 7 and $-6\sqrt{2}$ respectively
Obtain equations $x^2 - y^2 = 7$ and $2xy = -6\sqrt{2}$
Eliminate one variable and find an equation in the other [5]
Obtain $x^4 - 7x^2 - 18 = 0$ or $y^4 + 7y^2 - 18 = 0$, or 3-term equivalent
Obtain answers $\pm(3 - i\sqrt{2})$

(b) (i) Show point representing $1 + 2i$
Show circle with radius 1 and centre $1 + 2i$ [4]
Show a half line from the point representing 1
Show line making the correct angle with the real axis

(ii) State or imply the relevance of the perpendicular from $1 + 2i$ to the line [2]
Obtain answer $\sqrt{2} - 1$ (or 0.414)

13. M/J 16/P33/Q9

(i) *EITHER*: Multiply numerator and denominator of $\frac{u}{v}$ by $2 + i$, or equivalent
Simplify the numerator to $-5 + 5i$ or denominator to 5
Obtain final answer $-1 + I$
OR: Obtain two equations in x and y and solve for x or for y [3]
Obtain $x = -1$ or $y = 1$
Obtain final answer $-1 + I$

(ii) Obtain $u + v = 1 + 2i$
In an Argand diagram show points A, B, C representing u, v and $u + v$ respectively [4]
State that OB and AC are parallel
State that $OB = AC$

(iii) Carry out an appropriate method for finding angle AOB, e.g. find $\arg(u / v)$ [2]
Show sufficient working to justify the given answer $\frac{3}{4}\pi$

14. O/N 15/P32/Q9, O/N 15/P31/Q9

(i) Show u in a relatively correct position
Show u^* in a relatively correct position
Show $u^* - u$ in a relatively correct position [4]
State or imply that $OABC$ is a parallelogram

(ii) *EITHER*: Substitute for u and multiply numerator and denominator by $3 + i$, or equivalent
Simplify the numerator to $8 + 6i$ or the denominator to 10
Obtain final answer $\frac{4}{5}+\frac{3}{5}i$, or equivalent

OR: Substitute for u, obtain two equations in x and y and solve for x or for y
Obtain $x=\frac{4}{5}$ or $y=\frac{3}{5}$, or equivalent
Obtain final answer $\frac{4}{5}+\frac{3}{5}i$, or equivalent [3]

(iii) State or imply $\arg(u^*/u)=\tan^{-1}(\frac{3}{4})$
Substitute exact arguments in $\arg(u^*/u)=\arg u^* - \arg u$
Fully justify the given statement using exact values [3]

15. O/N 15/P33/Q9

(a) Either
Find w using conjugate of $1+3i$
Obtain $\frac{7-i}{5}$ or equivalent
Square $x+iy$ form to find w^2
Obtain $w^2=\frac{48-14i}{25}$ and confirm modulus is 2
Use correct process for finding argument of w^2
Obtain -0.284 radians or $-16.3°$

Or 1
Find w using conjugate of $1+3i$
Obtain $\frac{7-i}{5}$ or equivalent
Find modulus of w and hence of w^2
Confirm modulus is 2
Find argument of w and hence of w^2
Obtain -0.284 radians or $-16.3°$

Or 2
Square both sides to obtain $(-8+6i)w^2=-12+16i$
Find w^2 using relevant conjugate
Use correct process for finding modulus of w^2
Confirm modulus is 2
Use correct process for finding argument of w^2
Obtain -0.284 radians or $-16.3°$

Or 3
Find modulus of LHS and RHS
Find argument of LHS and RHS
Obtain $\sqrt{10}\,e^{1.249i}\,w=\sqrt{20}\,e^{1.107i}$ or equivalent
Obtain $w=\sqrt{2}\,e^{-0.1419i}$ or equivalent
Use correct process for finding w^2
Obtain 2 and -0.284 radians or $-16.3°$

Or 4
Find moduli of $2+4i$ and $1+3i$
Obtain $\sqrt{20}$ and $\sqrt{10}$
Obtain $|w^2|=2$ correctly
Find $\arg(2+4i)$ and $\arg(1+3i)$
Use correct process for $\arg(w^2)$
Obtain -0.284 radians or $-16.3°$

Or 5 Let $w = a + ib$, form and solve simultaneous equations in a and b

$a = \frac{7}{5}$ and $b = -\frac{1}{5}$

Find modulus of w and hence of w^2

Confirm modulus is 2

Find argument of w and hence of w^2

Obtain -0.284 radians or $-16.3°$

Or 6 Find w using conjugate of $1+3i$

Obtain $\frac{7-i}{5}$ or equivalent

Use $|w^2| = w\overline{w}$

Confirm modulus is 2

Find argument of w and hence of w^2

Obtain -0.284 radians or $-16.3°$ [6]

(b) Draw circle with centre the origin and radius 5

Draw straight line parallel to imaginary axis in correct position

Use relevant trigonometry on a correct diagram to find argument(s)

Obtain $5e^{\pm\frac{1}{3}\pi i}$ or equivalents in required form [4]

16. M/J 15/P32/Q7

(i) Square $x + iy$ and equate real and imaginary parts to -1 and $4\sqrt{3}$

Obtain $x^2 - y^2 = -1$ and $2xy = 4\sqrt{3}$

Eliminate one unknown and find an equation in the other

Obtain $x^4 + x^2 - 12 = 0$ or $y^4 - y^2 - 12 = 0$, or three term equivalent

Obtain answers $\pm(\sqrt{3} + 2i)$ [5]

[If the equations are solved by inspection, give B2 for the answers and B1 for justifying them]

(ii) Show a circle with centre $-1 + 4\sqrt{3}$ in a relatively correct position

Show a circle with radius 1 and centre not at the origin

Carry out a complete method for calculating the greatest value of arg z

Obtain answer 1.86 or 106.4° [4]

17. M/J 15/P31/Q8

(i) Either Expand $(2-i)^2$ to obtain $3-4i$ or unsimplified equivalent

Multiply by $\frac{3+4i}{3+4i}$ and simplify to $x+iy$ form or equivalent

Confirm given answer $2+4i$

Or Expand $(2-i)^2$ to obtain $3-4i$ or unsimplified equivalent

Obtain two equations in x and y and solve for x or y

Confirm given answer $2+4i$ [3]

(ii) Identify $4+4$ or $-4+4i$ as point at either end or state $p=2$ or state $p=-6$

Use appropriate method to find both critical values of p [3]

State $-6 \leqslant p \leqslant 2$

(iii) Identify equation as of form $|z-a| = a$ or equivalent

Form correct equation for a not involving modulus, e.g. $(a-2)^2 + 4^2 = a^2$ [3]

State $|z-5| = 5$

18. M/J 15/P33/Q8

(i) *EITHER*: Substitute for u in $\frac{i}{u}$ and multiply numerator and denominator by $1 + i$

Obtain final answer $-\frac{1}{2}+\frac{1}{2}i$, or equivalent

OR: Substitute for u, obtain two equations in x and y and solve for x or for y

Obtain final answer $-\frac{1}{2}+\frac{1}{2}i$, or equivalent **2**

(ii) Show a point representing u in a relatively correct position
Show the bisector of the line segment joining u to the origin
Show a circle with centre at the point representing i
Show a circle with radius 2 **4**

(iii) State argument $-\frac{1}{2}\pi$, or equivalent, e.g. 270°

State or imply the intersection in the first quadrant represents $2 + i$
State argument 0.464, (0.4636)or equivalent, e.g. 26.6° (26.5625) **3**

19. O/N 14/P32/Q5, O/N 14/P31/Q5

(i) Substitute $z = 1 + i$ and obtain $w = \frac{1+2i}{1+i}$

EITHER: Multiply numerator and denominator by the conjugate of the denominator, or equivalent
Simplify numerator to $3 + i$ or denominator to 2

Obtain final answer $\frac{3}{2}+\frac{1}{2}i$, or equivalent

OR: Obtain two equations in x and y, and solve for x or for y

Obtain $x = \frac{3}{2}$ or $y = \frac{1}{2}$, or equivalent

Obtain final answer $\frac{3}{2}+\frac{1}{2}i$, or equivalent **[4]**

(ii) *EITHER*: Substitute $w = z$ and obtain a 3-term quadratic equation in z,
e.g. $iz^2 + z - i = 0$
Solve a 3-term quadratic for z or substitute $z = x + iy$ and use a correct method to solve for x and y

OR: Substitute $w = x + iy$ and obtain two correct equations in x and y by equating real and imaginary parts
Solve for x and y

Obtain a correct solution in any form, e.g. $z = \frac{-1 \pm \sqrt{3}\,i}{2i}$

Obtain final answer $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$ **[4]**

20. O/N 14/P33/Q5

(i) State or imply $iw = -3 + 5i$

Carry out multiplication by $\frac{4-i}{4-i}$

Obtain final answer $-\frac{7}{17}+\frac{23}{17}i$ or equivalent **[3]**

(ii) Multiply w by z to obtain $17+17\mathrm{i}$

State $\arg w=\tan^{-1}\frac{3}{5}$ or $\arg z=\tan^{-1}\frac{1}{4}$

State $\arg wz=\arg w+\arg z$ [4]

Confirm given result $\tan^{-1}\frac{3}{5}+\tan^{-1}\frac{1}{4}=\frac{1}{4}\pi$ legitimately

21. M/J 14/P32/Q7

(a) *EITHER*: Substitute and expand $(-1+\sqrt{5}\,\mathrm{i})^3$ completely

Use $\mathrm{i}^2=-1$ correctly at least once

Obtain $a=-12$

State that the other complex root is $-1-\sqrt{5}\,\mathrm{i}$

OR1: State that the other complex root is $-1-\sqrt{5}\,\mathrm{i}$

State the quadratic factor z^2+2z+6

Divide the cubic by a 3-term quadratic, equate remainder to zero and solve for a or, using a 3-term quadratic, factorise the cubic and determine a

Obtain $a=-12$

OR2: State that the other complex root is $-1-\sqrt{5\mathrm{i}}$

State or show the third root is 2

Use a valid method to determine a

Obtain $a=-12$

OR3: Substitute and use De Moivre to cube $\sqrt{6}\mathrm{cis}(114.1^\circ)$, or equivalent

Find the real and imaginary parts of the expression

Obtain $a=-12$ 4

State that the other complex root is $-1-\sqrt{5\mathrm{i}}$

(b) *EITHER:* Substitute $w=\cos 2\theta+\mathrm{i}\sin 2\theta$ in the given expression

Use double angle formulae throughout

Express numerator and denominator in terms of $\cos\theta$ and $\sin\theta$ only

Obtain given answer correctly

OR: Substitute $w=\mathrm{e}^{2\mathrm{i}\theta}$ in the given expression

Divide numerator and denominator by $\mathrm{e}^{\mathrm{i}\theta}$, or equivalent

Express numerator and denominator in terms of $\cos\theta$ and $\sin\theta$ only 4

Obtain the given answer correctly

22. M/J 14/P31/Q5

(i) Either Multiply numerator and denominator by $\sqrt{3}+\mathrm{i}$ and use $\mathrm{i}^2=-1$

Obtain correct numerator $18+18\sqrt{3}\mathrm{i}$ or correct denominator 4

Obtain $\frac{9}{2}+\frac{9}{2}\sqrt{3}\mathrm{i}$ or $\left(18+18\sqrt{3}\mathrm{i}\right)/4$

Obtain modulus or argument [5]

Obtain $9\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$

OR Obtain modulus and argument of numerator or denominator, or both moduli or both arguments

Obtain moduli and argument 18 and $\frac{1}{6}\pi$ or 2 and $-\frac{1}{6}\pi$

or moduli 18 and 2 or arguments $\frac{1}{6}\pi$ and $-\frac{1}{6}\pi$ (allow degrees)

Obtain $18\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}\div 2\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}$ or equivalent

Divide moduli and subtract arguments [5]

Obtain $9\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$

State $3e^{\frac{1}{6}\pi i}$, following through their answer to part (i)

(ii) State $3e^{\frac{1}{6}\pi i \pm \frac{1}{2}\pi i}$, following through their answer to part (i)

Obtain $3e^{-\frac{5}{6}\pi i}$ [3]

23. M/J 14/P33/Q7

(a) *EITHER*: Multiply numerator and denominator by $1-4i$, or equivalent, and use $i^2=-1$

Simplify numerator to $-17-17i$, or denominator to 17

Obtain final answer $-1-i$

OR: Using $i^2=-1$, obtain two equations in x and y, and solve for x or for y

Obtain $x=-1$ or $y=-1$, or equivalent

Obtain final answer $-1-i$ 3

(b) (i) Show a point representing $2+i$ in relatively correct position

Show a circle with centre $2+i$ and radius 1

Show the perpendicular bisector of the line segment joining i and 2

Shade the correct region 4

(ii) State or imply that the angle between the tangents from the origin to the circle is required

Obtain answer 0.927 radians (or 53.1°) 2

24. O/N 13/P32/Q8

(a) *EITHER:* Solve for u or for v

Obtain $u=\dfrac{2i-6}{1-2i}$ or $v=\dfrac{5}{1-2i}$, or equivalent

Either: Multiply a numerator and denominator by conjugate of denominator, or equivalent

Or: Set u or v equal to $x+iy$, obtain two equations by equating real and imaginary parts and solve for x or for y

OR: Using $a+ib$ and $c+id$ for u and v, equate real and imaginary parts and obtain four equations in a, b, c and d

Obtain $b+2d=2$, $a+2c=0$, $a+d=0$ and $-b+c=3$, or equivalent

Solve for one unknown

Obtain final answer $u=-2-2i$, or equivalent

Obtain final answer $v=1+2i$, or equivalent [5]

(b) Show a circle with centre $-i$

Show a circle with radius 1

Show correct half line from 2 at an angle of $\frac{3}{4}\pi$ to the real axis

Use a correct method for finding the least value of the modulus

Obtain final answer $\dfrac{3}{\sqrt{2}}-1$, or equivalent, e.g. 1.12 (allow 1.1) [5]

25. O/N 13/P33/Q9

(a) Solve using formula, including simplification under square root sign

Obtain $\dfrac{-2\pm 4i}{2(2-i)}$ or similarly simplified equivalents

Multiply by $\dfrac{2+i}{2+i}$ or equivalent in at least one case

Obtain final answer $-\dfrac{4}{5}+\dfrac{3}{5}i$

Obtain final answer $-i$ [5]

(b) Show w in first quadrant with modulus and argument relatively correct
Show w^3 in second quadrant with modulus and argument relatively correct
Show w^* in fourth quadrant with modulus and argument relatively correct
Use correct method for area of triangle
Obtain 10 by calculation [5]

26. M/J 13/P32/Q9

(a) Substitute $w = x + iy$ and state a correct equation in x and y
Use $i^2 = -1$ and equate real parts
Obtain $y = -2$ [5]
Equate imaginary parts and solve for x
Obtain $x = 2\sqrt{2}$, or equivalent, only

(b) Show a circle with centre 2i
Show a circle with radius 2
Show half line from -2 at $\frac{1}{4}\pi$ to real axis
Shade the correct region [6]
Carry out a complete method for calculating the greatest value of $|z|$
Obtain answer 3.70

27. M/J 13/P31/Q7

(a) State or imply $3a + 3bi + 2i(a - bi) = 17 + 8i$
Consider real and imaginary parts to obtain two linear equations in a and b [4]
Solve two simultaneous linear equations for a or b
Obtain $7 - 2i$

(b) Either Show or imply a triangle with side 2
State at least two of the angles $\frac{1}{4}\pi, \frac{2}{3}\pi$ and $\frac{1}{12}\pi$
State or imply argument is $\frac{1}{4}\pi$
Use sine rule or equivalent to find r
Obtain $6.69e^{\frac{1}{4}\pi i}$

Or State $y = x$.
State $y = \frac{1}{\sqrt{3}}x + 2$ or $\frac{\sqrt{3}}{2} = \frac{x}{\sqrt{x^2 + (y-2)^2}}$ or $\frac{1}{2} = \frac{y-2}{\sqrt{x^2 + (y-2)^2}}$
State or imply argument is $\frac{\pi}{4}$
Solve for x or y. [5]
Obtain $6.69e^{\frac{1}{4}\pi i}$

28. M/J 13/P33/Q7

(i) Show that $a^2 + b^2 = (a + ib)(a - ib)$ [2]
Show that $(a + ib - ki)^* = a - ib + ki$

(ii) Square both sides and express the given equation in terms of z and z^*
Obtain a correct equation in any form, e.g. $(z - 10i)(z^* + 10i) = 4(z - 4i)(z^* + 4i)$
Obtain the given equation
Either express $|z - 2i| = 4$ in terms of z and z^* or reduce the given equation to the form
$|z - u| = r$ [5]
Obtain the given answer correctly [1]

(iii) State that the locus is a circle with centre 2i and radius 5

29. O/N 12/P33/Q10

(a) Expand and simplify as far as $iw^2 = -8i$ or equivalent
Obtain first answer $i\sqrt{8}$, or equivalent
Obtain second answer $-i\sqrt{8}$, or equivalent and no others [3]

(b) (i) Draw circle with centre in first quadrant
Draw correct circle with interior shaded or indicated [2]

(ii) Identify ends of diameter corresponding to line through origin and centre
Obtain $p = 3.66$ and $q = 7.66$
Show tangents from origin to circle
Evaluate $\sin^{-1}\left(\frac{1}{4}\sqrt{2}\right)$
Obtain $\alpha = \frac{1}{4}\pi - \sin^{-1}\left(\frac{1}{4}\sqrt{2}\right)$ or equivalent and hence 0.424
Obtain $\beta = \frac{1}{4}\pi + \sin^{-1}\left(\frac{1}{4}\sqrt{2}\right)$ or equivalent and hence 1.15 [6]

30. M/J 12/P32/Q7

(i) *EITHER*: Multiply numerator and denominator by $1 + 3i$, or equivalent
Simplify numerator to $-5 + 5i$, or denominator to 10, or equivalent
Obtain final answer $-\frac{1}{2} + \frac{1}{2}i$, or equivalent

OR: Obtain two equations in x and y, and solve for x or for y
Obtain $x = -\frac{1}{2}$ or $y = \frac{1}{2}$, or equivalent
Obtain final answer $-\frac{1}{2} + \frac{1}{2}i$, or equivalent [3]

(ii) Show B and C in relatively correct positions in an Argand diagram
Show u in a relatively correct position [2]

(iii) Substitute exact arguments in the LHS $\arg(1 + 2i) - \arg(1 - 3i) = \arg u$, or equivalent
Obtain and use $\arg u = \frac{3}{4}\pi$
Obtain the given result correctly [3]

31. M/J 12/P31/Q4

(i) Either Expand $(1 + 2i)^2$ to obtain $-3 + 4i$ or unsimplified equivalent
Multiply numerator and denominator by $2 - i$
Obtain correct numerator $-2 + 11i$ or correct denominator 5
Obtain $-\frac{2}{5} + \frac{11}{5}i$ or equivalent

Or Expand $(1 + 2i)^2$ to obtain $-3 + 4i$ or unsimplified equivalent
Obtain two equations in x and y and solve for x or y
Obtain final answer $x = -\frac{2}{5}$
Obtain final answer $y = \frac{11}{5}$ [4]

(ii) Draw a circle
Show centre at relatively correct position, following their u
Draw circle passing through the origin [3]

32. M/J 12/P33/Q10

(a) EITHER: Eliminate u or w and obtain an equation in w or in u

Obtain a quadratic in u or w, e.g. $u^2 - 4iu - 5 = 0$ or $w^2 + 4iw - 5 = 0$

Solve a 3-term quadratic for u or for w

OR1: Having squared the first equation, eliminate u or w and obtain an equation in w or u

Obtain a 2-term quadratic in u or w, e.g. $u^2 = -3 + 4i$

Solve a 2-term quadratic for u or for w

OR2: Using $u = a + ib$, $w = c + id$, equate real and imaginary parts and obtain 4 equations in a, b, c and d

Obtain 4 correct equations

Solve for a and b, or for c and d

Obtain answer $u = 1 + 2i$, $w = 1 - 2i$ [5]

Obtain answer $u = -1 + 2i$, $w = -1 - 2i$ and no other

(b) (i) Show point representing $2 - 2i$ in relatively correct position

Show a circle with centre $2 - 2i$ and radius 2

Show line for $\arg z = -\frac{1}{4}\pi$

Show line for $\text{Re } z = 1$ [5]

Shade the relevant region [1]

(ii) State answer $2 + \sqrt{2}$, or equivalent (accept 3.41)

33. O/N 11/P32/Q10, O/N 11/P31/Q10

(a) *EITHER*: Square $x + iy$ and equate real and imaginary parts to 1 and $-2\sqrt{6}$ respectively

Obtain $x^2 - y^2 = 1$ and $2xy = -2\sqrt{6}$

Eliminate one variable and find an equation in the other

Obtain $x^4 - x^2 - 6 = 0$ or $y^4 + y^2 - 6 = 0$, or 3-term equivalent [5]

Obtain answers $\pm(\sqrt{3} - i\sqrt{2})$

OR: Denoting $1 - 2\sqrt{6}i$ by $R\text{cis}\theta$, state, or imply, square roots are $\pm\sqrt{R}\text{cis}(\frac{1}{2}\theta)$ and find values of R and either $\cos\theta$ or $\sin\theta$ or $\tan\theta$

Obtain $\pm\sqrt{5}(\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta)$, and $\cos\theta = \frac{1}{5}$ or $\sin\theta = -\frac{2\sqrt{6}}{5}$ or $\tan\theta = -2\sqrt{6}$

Use correct method to find an exact value of $\cos\frac{1}{2}\theta$ or $\sin\frac{1}{2}\theta$

Obtain $\cos\frac{1}{2}\theta = \pm\sqrt{\frac{3}{5}}$ and $\sin\frac{1}{2}\theta = \pm\sqrt{\frac{2}{5}}$, or equivalent

Obtain answers $\pm(\sqrt{3} - i\sqrt{2})$, or equivalent

[Condone omission of $\pm$ except in the final answers.]

(b) Show point representing $3i$ on a sketch of an Argand diagram

Show a circle with centre at the point representing $3i$ and radius 2

Shade the interior of the circle

Carry out a complete method for finding the greatest value of arg z [5]

Obtain answer 131.8° or 2.30 (or 2.3) radians

[The f.t. is on solutions where the centre is at the point representing $-3i$.]

34. O/N 11/P33/Q6

(i) Use correct method for finding modulus of their w^2 or w^3 or both

Obtain $|w^2| = 2$ and $|w^3| = 2\sqrt{2}$ or equivalent

Use correct method for finding argument of their w^2 or w^3 or both

Obtain $\arg(w^2) = -\frac{1}{2}\pi$ or $\frac{3}{2}\pi$ and $\arg(w^3) = \frac{1}{4}\pi$ [4]

(ii) Obtain centre $-\frac{1}{2}-\frac{1}{2}i$ (their w^2)
Calculate the diameter or radius using $|w-w^2|$ w21 or right-angled triangle or cosine rule or equivalent
Obtain radius $\frac{1}{2}\sqrt{10}$ or equivalent
Obtain $\left|z+\frac{1}{2}+\frac{1}{2}i\right| = \frac{1}{2}\sqrt{10}$ or equivalent [4]

35. M/J 11/P32/Q7

(a) (i) *EITHER*: Multiply numerator and denominator by $a-2i$, or equivalent
Obtain final answer $\frac{5a}{a^2+4}-\frac{10i}{a^2+4}$, or equivalent
OR: Obtain two equations in x and y, solve for x or for y
Obtain final answer $x=\frac{5a}{a^2+4}$ and $y=\frac{10}{a^2+4}$, or equivalent [2]

(ii) Either state $\arg(u) = -\frac{3}{4}\pi$, or express u^* in terms of a (f.t. on u)
Use correct method to form an equation in a, e.g. $5a = -10$
Obtain $a = -2$ correctly [3]

(b) Show a point representing $2+2i$ in relatively correct position in an Argand diagram
Show the circle with centre at the origin and radius 2
Show the perpendicular bisector of the line segment from the origin to the point representing $2+2i$
Shade the correct region [4]
[SR: Give the first B1 and the B1√ for obtaining $y = 2 - x$, or equivalent, and sketching the attempt.]

36. M/J 11/P31/Q8

(i) Either: Multiply numerator and denominator by $(1-2i)$, or equivalent
Obtain $-3i$
State modulus is 3
Refer to u being on negative imaginary axis or equivalent and confirm argument as $-\frac{1}{2}\pi$
Or: Using correct processes, divide moduli of numerator and denominator
Obtain 3
Subtract argument of denominator from argument of numerator
Obtain $-\tan^{-1}\frac{1}{2}-\tan^{-1}2$ or $-0.464-1.107$ and hence $-\frac{1}{2}\pi$ or -1.57 [4]

(ii) Show correct half-line from u at angle $\frac{1}{4}\pi$ to real direction
Use correct trigonometry to find required value
Obtain $\frac{3}{2}\sqrt{2}$ or equivalent [3]

(iii) Show, or imply, locus is a circle with centre $(1+i)u$ and radius 1
Use correct method to find distance from origin to furthest point of circle
Obtain $3\sqrt{2}+1$ or equivalent [3]

37. M/J 11/P33/Q7

(i) Use the quadratic formula, completing the square, or the substitution $z = x + iy$ to find a root and use $i^2 = -1$
Obtain final answers $-\sqrt{3}\pm i$, or equivalent [2]

(ii) State that the modulus of both roots is 2
State that the argument of $-\sqrt{3}+i$ is 150° or $\frac{5}{6}\pi$ (2.62) radians
State that the argument of $-\sqrt{3}-i$ is −150° (or 210°) or $-\frac{5}{6}\pi$ (−2.62) radians or $\frac{7}{6}\pi$ (3.67) radians [3]

(iii) Carry out an attempt to find the sixth power of a root
Verify that one of the roots satisfies $z^6 = -64$ [3]
Verify that the other root satisfies the equation

38. O/N 10/P32/Q6

(i) State modulus is 2 [2]
State argument is $\frac{1}{6}\pi$, or 30°, or 0.524 radians

(ii) **(a)** State answer $3\sqrt{3}+i$
(b) *EITHER*: Multiply numerator and denominator by $\sqrt{3}-i$, or equivalent
Simplify denominator to 4 or numerator to $2\sqrt{3}+2i$
Obtain final answer $\frac{1}{2}\sqrt{3}+\frac{1}{2}i$, or equivalent
OR 1: Obtain two equations in x and y and solve for x or for y
Obtain $x=\frac{1}{2}\sqrt{3}$ or $y=\frac{1}{2}$
Obtain final answer $\frac{1}{2}\sqrt{3}+\frac{1}{2}i$, or equivalent
OR 2: Using the correct processes express iz^*/z in polar form
Obtain $x=\frac{1}{2}\sqrt{3}$ or $y=\frac{1}{2}$ [4]
Obtain final answer $\frac{1}{2}\sqrt{3}+\frac{1}{2}i$, or equivalent

(iii) Plot A and B in relatively correct positions
EITHER: Use fact that angle $AOB = \arg(iz^*) - \arg z$
Obtain the given answer
OR 1: Obtain $\tan A\hat{O}B$ from gradients of OA and OB and the correct $\tan(A-B)$ formula
Obtain the given answer
OR 2: Obtain $\cos A\hat{O}B$ by using correct cosine formula or scalar product [3]
Obtain the given answer

39. O/N 10/P33/Q3

(i) Attempt multiplication and use $i^2 = -1$ [3]
Obtain $3+4i$
Obtain 5 for <u>modulus</u>

(ii) Draw complete circle with centre corresponding to their w^2 … [3]
… and radius corresponding to their $|w^2|$
Shade the correct region

40. M/J 10/P32/Q8

(i) *EITHER*: State a correct expression for $|z|$ or $|z|^2$, e.g. $(1+\cos 2\theta)^2 + (\sin 2\theta)^2$
Use double angle formulae throughout or Pythagoras
Obtain given answer $2\cos\theta$ correctly
State a correct expression for tangent of argument, e.g. $(\sin 2\theta/(1+\cos 2\theta)$
Use double angle formulae to express it in terms of $\cos\theta$ and $\sin\theta$
Obtain $\tan\theta$ and state that the argument is θ

OR: Use double angle formulae to express z in terms of $\cos\theta$ and $\sin\theta$
Obtain a correct expression, e.g. $1+\cos^2\theta-\sin^2\theta+2i\sin\theta\cos\theta$
Convert the expression to polar form
Obtain $2\cos\theta(\cos\theta+i\sin\theta)$
State that the modulus is $2\cos\theta$
State that the argument is θ [6]

(ii) Substitute for z and multiply numerator and denominator by the conjugate of z, or equivalent
Obtain correct real denominator in any form
Identify and obtain real part equal to $\frac{1}{2}$ [3]

41. M/J 10/P31/Q7

(i) Obtain modulus $\sqrt{8}$
Obtain argument $\frac{1}{4}\pi$ or 45° [2]

(ii) Show 1, i and u in relatively correct positions on an Argand diagram
Show the perpendicular bisector of the line joining 1 and i
Show a circle with centre u and radius 1
Shade the correct region [4]

(iii) State or imply relevance of the appropriate tangent from O to the circle
Carry out complete strategy for finding $|z|$ for the critical point
Obtain answer $\sqrt{7}$ [3]

42. M/J 10/P33/Q8

(a) *EITHER*: Substitute $1+i\sqrt{3}$, attempt complete expansions of the x^3 and x^2 terms
Use $i^2=-1$ correctly at least once
Complete the verification correctly
State that the other root is $1-i\sqrt{3}$

OR1: State that the other root is $1-i\sqrt{3}$
State quadratic factor x^2-2x+4
Divide cubic by 3-term quadratic reaching partial quotient $2x+k$
Complete the division obtaining zero remainder

OR2: State factorisation $(2x+3)(x^2-2x+4)$, or equivalent
Make reasonable solution attempt at a 3-term quadratic and use $i^2=-1$
Obtain the root $1+i\sqrt{3}$
State that the other root is $1-i\sqrt{3}$ [4]

(b) Show point representing $1+i\sqrt{3}$ in relatively correct position on an Argand diagram
Show circle with centre at $1+i\sqrt{3}$ and radius 1
Show line for $\arg z=\frac{1}{3}\pi$ making $\frac{1}{3}\pi$ with the real axis
Show line from origin passing through centre of circle, or the diameter which would contain the origin if produced
Shade the relevant region [5]

43. O/N 09/P32/Q7

(i) (a) State that $u+v$ is equal to $1+2i$ [1]

(b) *EITHER*: Multiply numerator and denominator of u/v by $3-i$, or equivalent
Simplify numerator to $-5+5i$, or denominator to 10
Obtain answer $-\frac{1}{2}+\frac{1}{2}i$, or equivalent

OR1: Obtain two equations in x and y and solve for x or for y
Obtain $x = -\frac{1}{2}$ or $y = \frac{1}{2}$
Obtain answer $-\frac{1}{2} + \frac{1}{2}$i, or equivalent

OR2: Using the correct processes express u/v in polar form
Obtain $x = -\frac{1}{2}$ or $y = \frac{1}{2}$ correctly
Obtain answer $-\frac{1}{2} + \frac{1}{2}$i, or equivalent [3]

(ii) State that the argument of u/v is $\frac{3}{4}\pi$ (2.36 radians or 135°) [1]

(iii) *EITHER*: Use facts that angle $AOB = \arg u - \arg v$ and $\arg u - \arg v = \arg(u/v)$
Obtain given answer

OR1: Obtain tan $A\hat{O}B$ from gradients of OA and OB and the tan $(A \pm B)$ formula
Obtain given answer

OR2: Obtain cos $A\hat{O}B$ by using the cosine formula or scalar product
Obtain given answer [2]

(iv) State $OA = BC$
State OA is parallel to BC [2]

44. O/N 09/P31/Q7

(i) Substitute $x = -2 + \text{i}$ in the equation and attempt expansion of $(-2 + \text{i})^3$
Use $\text{i}^2 = -1$ correctly at least once and solve for k
Obtain $k = 20$ [3]

(ii) State that the other complex root is $-2 - \text{i}$ [1]

(iii) Obtain modulus $\sqrt{5}$
Obtain argument 153.4° or 2.68 radians [2]

(iv) Show point representing u in relatively correct position in an Argand diagram
Show vertical line through $z = 1$
Show the correct half-lines from u of gradient zero and 1
Shade the relevant region [4]

[SR: For parts **(i)** and **(ii)** allow the following alternative method:
State that the other complex root is $-2 - \text{i}$
State quadratic factor $x^2 + 4x + 5$
Divide cubic by 3-term quadratic, equate remainder to zero and solve for k, or, using 3-term quadratic, factorise cubic and obtain k
Obtain $k = 20$

45. M/J 09/P03/Q7

(i) Use quadratic formula, or completing the square, or the substitution $z = x + \text{i}y$ to find a root, using $\text{i}^2 = -1$
Obtain a root, e.g. $1 - \sqrt{3}\text{i}$
Obtain the other root, e.g. $-1 - \sqrt{3}\text{i}$ 3

(ii) Represent both roots on an Argand diagram in relatively correct positions 1

(iii) State modulus of both roots is 2
State argument of $1 - \sqrt{3}\text{i}$ is $-60°$ (or $300°$, $-\frac{1}{3}\pi$, $\frac{5}{3}\pi$)
State argument of $-1 - \sqrt{3}\text{i}$ is $-120°$ (or $240°$, $-\frac{2}{3}\pi$, $\frac{4}{3}\pi$) 3

(iv) Give a complete justification of the statement 1

[The A marks in **(i)** are for the final versions of the roots. Allow $(\pm 2 - 2\sqrt{3}\text{i})/2$ as final answer. The remaining marks are only available for roots such that $xy \neq 0$.]
[Treat answers to **(iii)** in polar form as a misread]

46. O/N 08/P03/Q10

(i) State that the modulus of w is 1
State that the argument of w is $\frac{2}{3}\pi$ or 120° (accept 2.09, or 2.1) [2]

(ii) State that the modulus of wz is R
State that the argument of wz is $\theta + \frac{2}{3}\pi$
State that the modulus of z/w is R
State that the argument of z/w is $\theta - \frac{2}{3}\pi$ [4]

(iii) State or imply the points are equidistant from the origin
State or imply that two pairs of points subtend $\frac{2}{3}\pi$ at the origin, or that all three pairs subtend equal angles at the origin [2]

(iv) Multiply $4 + 2i$ by w and use $i^2 = -1$
Obtain $-(2+\sqrt{3}) + (2\sqrt{3}-1)i$, or exact equivalent
Divide $4 + 2i$ by w, multiplying numerator and denominator by the conjugate of w, or equivalent
Obtain $-(2-\sqrt{3}) - (2\sqrt{3}+1)i$, or exact equivalent [4]
[Use of polar form of $4 + 2i$ can earn M marks and then A marks for obtaining exact $x + iy$ answers.]
[SR: If answers only seen in polar form, allow B1+B1 in (i), B1√ + B1√ in (ii), but A0 + A0 in (iv).]

47. M/J 08/P03/Q5

(i) Find modulus of $2\cos\theta - 2i\sin\theta$ and show it is equal to 2
Show a circle with centre at the point representing **i**
Show a circle with radius 2 [3]

(ii) Substitute for z and multiply numerator and denominator by the conjugate of $z + 2 - i$, or equivalent
Obtain correct real denominator in any form
Identify and obtain correct unsimplified real part in terms of $\cos\theta$,
e.g. $(2\cos\theta + 2)/(8\cos\theta + 8)$ [4]
State that real part equals $\frac{1}{4}$

48. O/N 07/P03/Q8

(a) (i) *EITHER*: Carry out multiplication of numerator and denominator by $1 + 2i$, or equivalent
Obtain answer $2 + i$, or any equivalent of the form $(a + ib)/c$
*OR*1: Obtain two equations in x and y, and solve for x or for y
Obtain answer $2 + i$, or equivalent
*OR*2: Using the correct processes express z in polar form
Obtain answer $2 + i$, or equivalent [2]

(ii) State that the modulus of z is $\sqrt{5}$ or 2.24
State that the argument of z is 0.464 or 26.6° [2]

(b) *EITHER*: Square $x + iy$ and equate real and imaginary parts to 5 and -12 respectively
Obtain $x^2 - y^2 = 5$ and $2xy = -12$
Eliminate one variable and obtain an equation in the other
Obtain $x^4 - 5x^2 - 36 = 0$ or $y^4 + 5y^2 - 36 = 0$, or 3-term equivalent
Obtain answer $3 - 2i$
Obtain second answer $-3 + 2i$ and no others
[SR: Allow a solution with $2xy = 12$ to earn the second A1 and thus a maximum of 3/6.]
OR: Convert $5 - 12i$ to polar form (R, θ)
Use the fact that a square root has the polar form $(\sqrt{R}, \frac{1}{2}\theta)$
Obtain one root in polar form, e.g. $(\sqrt{13}, -0.588)$ or $(\sqrt{13}, -33.7°)$
Obtain answer $3 - 2i$
Obtain answer $-3 + 2i$ and no others [6]

49. M/J 07/P03/Q8

(i) *EITHER*: Carry out multiplication of numerator and denominator by $-1-i$, or solve for x or y
Obtain $u=-1-i$, or any equivalent of the form $(a+ib)/c$
State modulus of u is $\sqrt{2}$ or 1.41
State argument of u is $-\frac{3}{4}\pi(-2.36)$ or $-135°$, or $\frac{5}{4}\pi(3.93)$ or $225°$

OR: Divide the modulus of the numerator by that of the denominator
State modulus of u is $\sqrt{2}$ or 1.41
Subtract the argument of the denominator from that of the numerator, or equivalent
State argument of u is $-\frac{3}{4}\pi(-2.36)$ or $-135°$, or $\frac{5}{4}\pi(3.93)$ or $225°$ [6]

Carry out method for finding the modulus or the argument of u^2
State modulus of u is 2 and argument of u^2 is $\frac{1}{2}\pi(1.57)$ or $90°$

(ii) Show u and u^2 in relatively correct positions
Show a circle with centre at the origin and radius 2
Show the line which is the perpendicular bisector of the line joining u and u^2 [4]
Shade the correct region, having obtained u and u^2 correctly

50. O/N 06/P03/Q9

(i) *EITHER*: Multiply numerator and denominator by $2+i$, or equivalent
Simplify numerator to $5+5i$ or denominator to 5
Obtain answer $1+i$

OR: Obtain two equations in x and y, and solve for x or for y
Obtain $x=1$
Obtain $y=1$

OR: Using correct processes express u in polar form [3]
Obtain $u=\sqrt{2}(\cos 45°+i\sin 45°)$, or equivalent
Obtain answer $1+i$ [2]

(ii) State that the modulus is $\sqrt{2}$ or 1.41
State that the argument is $45°$ or $\frac{1}{4}\pi$ (or 0.785)

(iii) Show the point representing u in a relatively correct position [3]
Show a circle with centre at the point representing u
Indicate or imply the radius is 1
[NB: If the Argand diagram has unequal scales the locus is not circular in appearance, but an ellipse with centre u and equal axes parallel to the axes of the diagram earns B1√ and B1 if both semi-axes are indicated or implied to be equal to 1. In such a situation only award B1√ for a circle with centre u and a horizontal or vertical radius indicated or implied to be 1.]

(iv) Carry out complete strategy for calculating min $|z|$ for the locus [2]
Obtain answer $\sqrt{2}-1$ (or 0.414)
[The f.t. is on the value of u.]

51. M/J 06/P03/Q7

(i) Show u and u^* in relatively correct positions
Show $u+u^*$ in relatively correct position
State or imply that $OACB$ is a parallelogram
State or imply that $OACB$ has a pair of adjacent equal sides [4]
[The statement that OACB is a rhombus, or equivalent, earns B2√]

(ii) *EITHER*: Multiply numerator and denominator of $\frac{u}{u^*}$ by $2+i$
Simplify numerator to $3+4i$ or denominator to 5
Obtain answer $\frac{3}{5}+\frac{4}{5}i$, or equivalent

OR: Obtain two equations in x and y, and solve for x or for y
Obtain $x = \frac{3}{5}$ or $y = \frac{4}{5}$
Obtain answer $\frac{3}{5} + \frac{4}{5}\text{i}$ [3]

(iii) *EITHER*: State or imply $\arg\left(\frac{u}{u^*}\right) = 2 \arg u$
Justify the given statement correctly
OR: Use tan 2A formula with $\tan A = \frac{1}{2}$
Justify the given statement correctly [2]
[The f.t. is on $-2 + \text{i}$ as complex conjugate]

52. O/N 05/P03/Q7

(i) Substitute $x = 1 + 2\text{i}$ and attempt expansions
Use $\text{i}^2 = -1$ correctly at least once
Complete the verification correctly [3]

(ii) State that the other complex root is $1 - 2\text{i}$ [1]

(iii) Show $1 + 2\text{i}$ in relatively correct position
Sketch a locus which
(a) is a straight line
(b) relative to the point representing $1 + 2\text{i}$ (call it A), passes through the mid-point of OA
(c) intersects OA at right angles [4]

53. M/J 05/P03/Q3

(i) Use quadratic formula, or the method of completing the square, or the substitution $z = x + \text{i}y$ to find a root, using $\text{i}^2 = -1$
Obtain a root, e.g. $2 + \text{i}$
Obtain the other root $-2 + \text{i}$ 3
[Roots given as $\pm 2 + \text{i}$ earn **A1** + **A1**.]

(ii) Obtain modulus $\sqrt{5}$ (or 2.24) of both roots
Obtain argument of $2 + \text{i}$ as 26.6° or 0.464 radians
(allow ±1 in final figure)
Obtain argument of $-2 + \text{i}$ as 153.4° or 2.68 radians
(allow ±1 in final figure) 3
[SR: in applying the follow through to the roots obtained in **(i)**, if both roots are real or pure imaginary, the mark for the moduli is not available and only **B1**√ is given if both arguments are correct; also if one of the two roots is real or pure imaginary and the other is neither then **B1**√ is given if both moduli are correct and **B1**√ if both arguments are correct.]

(iii) Show both roots on an Argand diagram in relatively correct positions 1
[This follow through is only available if at least one of the two roots is of the form $x + \text{i}y$ where $xy \neq 0$.]

54. O/N 04/P03/Q6

(i) State $u - v$ is $-3 + \text{i}$ B1
EITHER: Carry out multiplication of numerator and denominator of u/v by $4 - 2\text{i}$, or equivalent M1
Obtain answer $\frac{1}{2} + \frac{1}{2}\text{i}$, or any equivalent A1
OR: Obtain two equations in x and y, and solve for x or for y M1
Obtain answer $\frac{1}{2} + \frac{1}{2}\text{i}$, or any equivalent A1 3

(ii) State argument is $\frac{1}{4}\pi$ (or 0.785 radians or 45°) 1

(iii) State that *OC* and *BA* are equal (in length)
State that *OC* and *BA* are parallel or have the same direction 2

(iv) *EITHER*: Use fact that angle *AOB* = arg *u* – arg *v* = arg(*u*/*v*)
Obtain given answer (or 45°)

OR: Obtain tan *AOB* from gradients of *OA* and *OB* and the tan(*A* ± *B*) formula
Obtain given answer (or 45°)

OR: Obtain cos *AOB* by using the cosine rule or a scalar product
Obtain given answer (or 45°)

OR: Prove angle *OAB* = 90° and *OA* = *AB* 2
Derive the given answer (or 45°)

[SR: Obtaining a value for angle *AOB* by calculating $\arctan(3) - \arctan\left(\frac{1}{2}\right)$ earns a maximum of B1.]

55. M/J 04/P03/Q8

(i) *EITHER*: Solve the quadratic and use $\sqrt{-1} = i$
Obtain roots $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\frac{1}{2} - i\frac{\sqrt{3}}{2}$ or equivalent

OR: Substitute $x + iy$ and solve for x or y 2
Obtain correct roots

(ii) State that the modulus of each root is equal to 1 3
State that the arguments are $\frac{1}{3}\pi$ and $-\frac{1}{3}\pi$ respectively
[Accept degrees and $\frac{5}{3}\pi$ instead of $-\frac{1}{3}\pi$. Accept a modulus in the form $\sqrt{\frac{p}{q}}$ or $\sqrt{n}$, where *p*, *q*, *n* are integers. An answer which only gives roots in modulus-argument form earns B1 for both the implied moduli and B1 for both the implied arguments.]

(iii) *EITHER*: Verify $z^3 = -1$ for each root

OR: State $z^3 + 1 = (z+1)(z^2 - z + 1)$
Justify the given statement

OR: Obtain $z^3 = z^2 - z$ 2
Justify the given statement

56. O/N 03/P03/Q7

(i) *EITHER*: Attempt multiplication of numerator and denominator by 3 + 2i, or equivalent
Simplify denominator to 13 or numerator to 13 + 26i
Obtain answer $u = 1 + 2i$

OR: Using correct processes, find the modulus and argument of *u*
Obtain modulus $\sqrt{5}$ (or 2.24) or argument $\tan^{-1} 2$ (or 63.4° or 1.11 radians)
Obtain answer $u = 1 + 2i$ [3]

(ii) Show the point U on an Argand diagram in a relatively correct position
Show a circle with centre U
Show a circle with radius consistent with 2 **[3]**

[f.t. on the value of u.]

(iii) State or imply relevance of the appropriate tangent from O to the circle
Carry out a complete strategy for finding max arg z
Obtain final answer 126.9° (2.21 radians) **[3]**

[Drawing the appropriate tangent is sufficient for B1√.]
[A final answer obtained by measurement earns M1 only.]

57. M/J 03/P03/Q5

(i) State or imply $w = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi$ (allow decimals)
Obtain answer $uw = -\sqrt{3} - i$ (allow decimals)
Multiply numerator and denominator of $\frac{u}{w}$ by $-1 - i\sqrt{3}$, or equivalent
Obtain answer $\frac{u}{w} = \sqrt{3} - i$ (allow decimals) **[4]**

(ii) Show U on an Argand diagram correctly
Show A and B in relatively correct positions **[2]**

(iii) Prove that $AB = UA$ (or UB), or prove that angle AUB = angle ABU (or angle BAU) or prove, for example, that $AO = OB$ and angle AOB = 120°, or prove that one angle of triangle UAB equals 60°
Complete a proof that triangle UAB is equilateral **[2]**

58. O/N 02/P03/Q8

(a) *EITHER*: Square $x + iy$ and real and/or imaginary parts to -3 and/or 4 respectively
Obtain $x^2 - y^2 = -3$ and $2xy = 4$
Eliminate one variable and obtain an equation in the other variable
Obtain $x^4 + 3x^2 - 4 = 0$, or $y^4 - 3y^2 - 4 = 0$, or 3-term equivalent
Obtain final answers $\pm(1 + 2i)$ and no others
[Accept $\pm1\pm2i$, or $x = 1, y = 2$ and $x = -1, y = -2$ as final answers, but not $x = \pm1, y = \pm2$.]

OR: Convert $-3 + 4i$ to polar form (R, θ)
Use fact that a square root has polar form $(\sqrt{R}, \frac{1}{2}\theta)$
Obtain one root in polar form e.g. $(\sqrt{5}, 63.4°)$ (allow 63.5°; argument is 1.11 radians)
Obtain anwer $1 + 2i$
Obtain answer $-1 - 2i$ and no others **[5]**

(b) (i) Carry out multiplication of numerator and denominator by 2+i
Obtain answer $\frac{1}{5} + \frac{7}{5}i$ or $0.2 + 1.4i$ **[2]**

(ii) Show all three points on an Argand diagram in relatively correct positions **[1]**
[Accept answers on separate diagrams.]

(iii) State that $OC = \frac{OA}{OB}$, or equivalent **[1]**
[Accept the answer $OA.OC = 2OB$, or equivalent.]
[Accept answers with $|OA|$ for OA etc.]